Lecture - 18

LMI and ARE

- Linear Matrix Inequality (LMI) takes the following form

$$F(x) = F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n$$

where x_i are variables (for $i = 1, 2 \cdots n$)

The matrices $F_k = F_k^T \in \mathbb{R}^{n \times n}$ for $k = 0, 1, \cdots n$, are given.

$F(x)$ is symmetric matrix & it is linear with variables x_i

$F(x) \succcurlyeq 0$, $F(x) \preccurlyeq 0$, $F(x) \succ 0$, $F(x) \prec 0$

$F(x) \succcurlyeq 0$ → symmetric positive semidefinite matrix → for all $z \in \mathbb{R}^n$

$$z^T F(x) z \geq 0$$

$F(x) \succ 0$ → symmetric positive definite matrix → for all $z \in \mathbb{R}^n$ $(z \neq 0)$

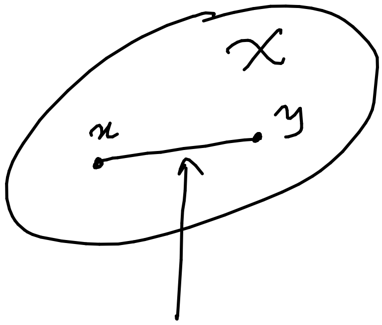
$$z^T F(x) z > 0$$

The solution set of an LMI :

$$\mathcal{X} ::= \{ x \mid \underline{F(x) > 0} \} \qquad F(x) \text{ is SPD} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\downarrow$

This set is a <u>CONVEX</u> set .



$\lambda x + (1-\lambda) y$
where $\lambda \in [0,1]$

x & y are sol^n of LMIs

Hence

$F(x) > 0 \qquad F(y) > 0$

Let us define the line segment $z = \lambda x + (1-\lambda) y$, $\lambda \in [0,1]$

$$F(z) = F(\lambda x + (1-\lambda) y) .$$

$$= F_0 + \sum_{i=1}^{n} (\lambda x_i + (1-\lambda) y_i) F_i .$$

$$= \lambda F_0 + (1-\lambda) F_0 + \sum_{i=1}^{n} (\lambda x_i + (1-\lambda) y_i) F_i .$$

$$= \lambda \left(F_0 + \sum_{i=1}^{n} x_i F_i \right) + (1-\lambda) \left(F_0 + \sum_{i=1}^{n} y_i F_i \right)$$

$$= \lambda F(x) + (1-\lambda) F(y)$$

Since $F(x) > 0$ & $F(y) > 0$ & $\lambda \in [0,1)$

$$F(z) > 0$$

This says that the sol^n set $\mathcal{X}$ is a convex set.

→ Two types of problems related to LMI

- LMI feasibility problem

Find x s.t

$F(x) \succ 0$ ← LMI

- LMI optimization problem

Convex optimization problem also called Semidefinite programs

$$\begin{cases} \min & c^T x \\ \text{s.t} & F(x) \succ 0 \end{cases}$$

(c is a given vector)

← LMI & its sol^n set is convex

→ The LMIs in different form

$A^T P + PA \succ 0$

is linear in variable $P = P^T$ (given A matrix)

→ Example, consider $P \in \mathbb{R}^{2\times 2}$

Define

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \& \quad P_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

P_i are symmetric

- All symmetric matrices of size 2×2 can be generated by using P_1, P_2 & P_3.

All possible 2×2 symmetric matrices can be characterized by

$$P(x) = x_1 P_1 + x_2 P_2 + x_3 P_3$$

$$A^T P + PA = A^T(x_1 P_1 + x_2 P_2 + x_3 P_3) + (x_1 P_1 + x_2 P_2 + x_3 P_3)A$$

$$= x_1 \underbrace{(A^T P_1 + P_1 A)}_{F_1} + x_2 \underbrace{(A^T P_2 + P_2 A)}_{F_2} + x_3 \underbrace{(A^T P_3 + P_3 A)}_{F_3}$$

$$= x_1 F_1 + x_2 F_2 + x_3 F_3$$

where F_i are symmetric matrix & also are given

Hence $A^T P + PA \succ 0$ is an LMI.

→ Another property

- If there are multiple LMIs

i.e.

$$\left.\begin{aligned} F_1(x) &\succ 0 \\ F_2(x) &\succ 0 \\ &\vdots \\ F_n(x) &\succ 0 \end{aligned}\right\}$$

soln set ? is also (convex) it is intersection of several convex sets

→ Let say we have the following situation

$$\left\{ \begin{array}{l} \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} > 0 \\ \quad \equiv \\ P > 0 \ \ \& \ \ R - Q^T P^{-1} Q > 0 \end{array} \right.$$

where P, Q, R are matrix variables.

P & R are symmetric matrices

Use transformation matrix $T = \begin{bmatrix} I & -P^{-1}Q \\ 0 & I \end{bmatrix}$

$$T^T \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} T = \begin{bmatrix} P & 0 \\ 0 & \underbrace{R - Q^T P^{-1} Q}_{\downarrow} \end{bmatrix}$$

Congruent transformation

Two problems

Constraint-1 $\begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} > 0$

Let the consts are in the form

Const-II $\left\{ \begin{array}{l} P > 0 \\ R - Q^T P^{-1} Q > 0 . \end{array} \right.$

Problem $\min \gamma$ s.t.

γ & P are variable

$$P > 0$$

$$A^T P + PA + C^T C + \frac{1}{\gamma} PBB^T P < 0$$

$\downarrow$ Convert to an LMI optimization problem

$\min \gamma$ s.t.

(i) $P > 0$

(ii) $$\begin{bmatrix} A^T P + PA + C^T C & PB \\ B^T P & -\gamma I \end{bmatrix} < 0$$

$$\rightarrow \quad A^T P + PA + PBR^{-1}B^T P + Q < 0$$

$$\downarrow$$

$$\begin{bmatrix} -A^T P - PA - Q & PB \\ B^T P & R \end{bmatrix} > 0$$

→ Algebraic Riccati Equations (ARE)

The AREs are of this form:

$$A^T X + XA + XR^{-1}X + Q = 0 \quad \cdots\cdots (*)$$

where $Q = Q^T \in \mathbb{R}^{n\times n}$ & $R = R^T \in \mathbb{R}^{n\times n}$

X is a variable

→ Let us define a matrix H corresponding to the ARE – $(*)$

$$\underbrace{H := \begin{bmatrix} A & R^{-1} \\ -Q & -A^T \end{bmatrix}}_{\mathbb{R}^{2n\times 2n}} \leftarrow \text{Hamiltonian matrix}$$

Property of H:

- If λ is an eigenvalue of H, then $-\lambda$ is also an eigenvalue of H.

Proof

Define a matrix $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}}_{J^{-1}} \underbrace{\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}}_{J} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\longrightarrow J^{-1} = -J$$

$$J^{-1}HJ = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} A & R^{-1} \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -A^T & Q \\ -R^{-1} & A \end{bmatrix} = -H^T$$

Hence H & $-H^T$ are similar matrices.

Let λ_i is an eigenvalue of H with eigenvector v_i

$$H v_i = \lambda_i v_i$$

$$H J J^{-1} v_i = \lambda_i v_i$$

$$\underbrace{J^{-1} H J}\, J^{-1} v_i = \lambda_i J^{-1} v_i$$

$$-H^T J^{-1} v_i = \lambda_i J^{-1} v_i$$

$$-H^T k_i = \lambda_i k_i \quad \text{where} \quad k_i = J^{-1} v_i$$

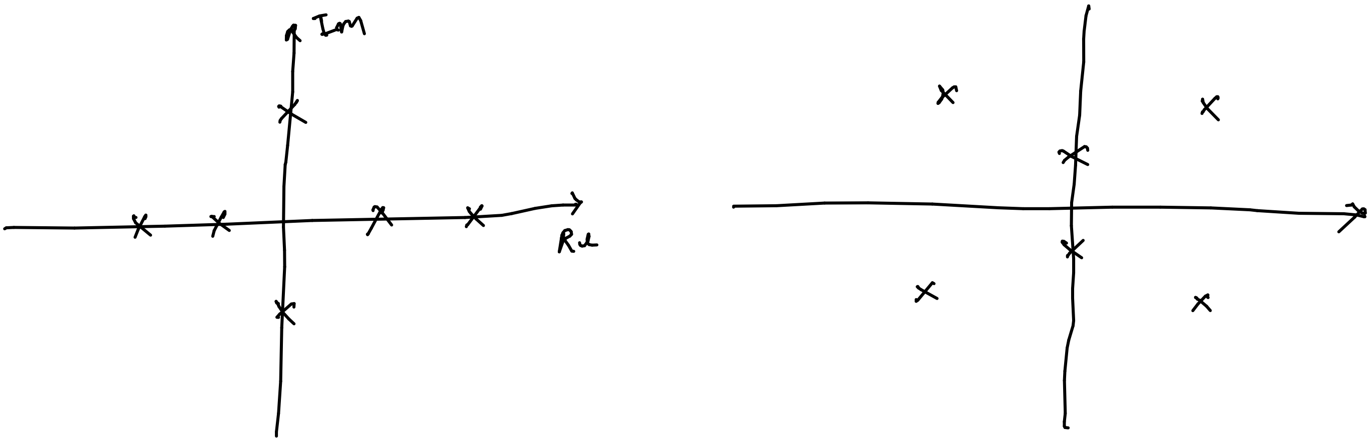
λ_i is an eigenvalue of $-H^T$ (with e.v. $J^{-1} v_i$)

The eigenvalue of H & H^T are same.

So the eigenvalue of $-H^T$ are $-\lambda_i$.

Hence if λ_i is an eigenvalue of H, then $-\lambda_i$ is also an eigenvalue of H.

Since H is real matrix, the eigenvalues are symmetric about real axis in the complex plane



Distribution of eigenvalues of H

The eigenvalues of H → belongs to the $\mathbb{C}^-$ (left half of $\mathbb{C}$)
→ belongs to $\mathbb{C}^0$ (on imaginary axis
→ belongs to $\mathbb{C}^+$ (Right hand side of $\mathbb{C}$

→ The sol^n of ARE

↳ Let us define two matrices

$X_1 \in \mathbb{R}^{n\times n}$ & $X_2 \in \mathbb{R}^{n\times n}$

such that $\mathrm{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is an

'n'-dimensional invariant subspace of H.

The 'n'-dimensional invariant subspace of H $\equiv \operatorname{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

Since $\operatorname{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is an n-dimensional invariant subspace of H, there exists Λ s.t.

$$\begin{bmatrix} A & R^{-1} \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\Lambda$$

$\left[\begin{array}{l} AV = V\Lambda \\ \Rightarrow V^{-1}AV = \Lambda \end{array}\right.$

$$\Rightarrow \begin{bmatrix} A & R^{-1} \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} X_1^{-1} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\Lambda X_1^{-1}$$

(Assume that X_1 is invertible)

$$\Rightarrow \begin{bmatrix} A & R^{-1} \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} I \\ X_2 X_1^{-1} \end{bmatrix} = \begin{bmatrix} I \\ X_2 X_1^{-1} \end{bmatrix} X_1 \Lambda X_1^{-1}$$

Let us denote $X = X_2 X_1^{-1}$

$$\Rightarrow \begin{bmatrix} A & R^{-1} \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} X_1 \Lambda X_1^{-1}$$

Premultiply both side by $[-X \;\; I]$

$$\begin{bmatrix} -X & I \end{bmatrix}\begin{bmatrix} A & R^{-1} \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} I \\ X \end{bmatrix} = 0$$

$$\Rightarrow A^T X + XA + XR^{-1}X + Q = 0$$

Here in $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ if X_1 is invertible,

then $\boxed{X = X_2 X_1^{-1}}$ is a solution of the ARE - ⊛ .

Example : $A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$ $R^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ $Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$H = \left[\begin{array}{cc|cc} -3 & 2 & 0 & 0 \\ -2 & 1 & 0 & -1 \\ \hline 0 & 0 & 3 & 2 \\ 0 & 0 & -2 & -1 \end{array}\right] \rightarrow \text{eigenvalues}$$

$1, 1, -1$ & -1

→ the e-vector & generalized eigenvectors corresponding to 1

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -1 \\ -3/2 \\ 1 \\ 0 \end{bmatrix}$$

the e. vectors & generalized eigenvectors corresponding to -1

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 3/2 \\ 0 \\ 0 \end{bmatrix}$$

Let us choose v_1 & v_2 to form $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

So $$\begin{bmatrix} X_1 \\ \hline X_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -3/2 \\ \hline 2 & 1 \\ -2 & 0 \end{bmatrix}$$

The sol^n of corresponding ARE is

$$X = X_2 X_1^{-1} = \begin{bmatrix} -10 & 6 \\ 6 & -4 \end{bmatrix}.$$

→ <u>Stabilizing Solution & Riccati Operator</u>

$$H \begin{cases} \text{EVs in } \mathbb{C}^+ \\ \text{EVs in } \mathbb{C}^0 \\ \text{EVs in } \mathbb{C}^- \end{cases}$$

We will look at H which does not have eigenvalues on imaginary axis.

Definition : Suppose that H is a Hamiltonian matrix. Then, H is said to be in the <u>domain of Riccati operator</u> if there exists square $n \times n$ matrices : H_-, X_1, X_2 s.t.

(D) ... $$\begin{cases} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_- = H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \text{ with } \quad (VA = \Lambda V \\ H_- \text{ is Hurwitz matrix and } X_1 \text{ is invertible.} \end{cases}$$

So if Δ holds, then

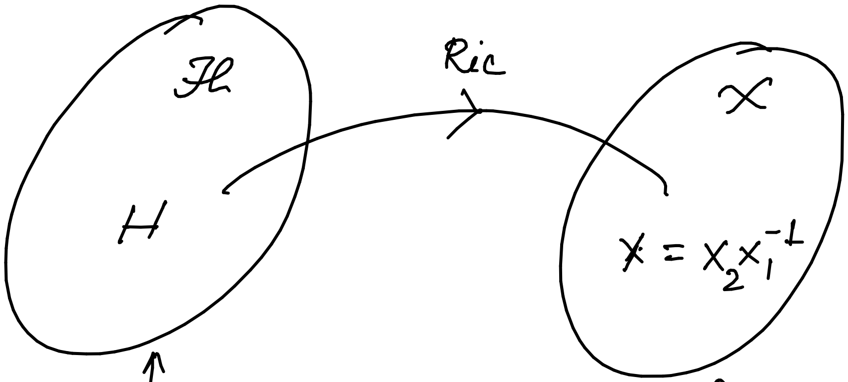
H has 'n' eigenvalues which are in the open left half of $\mathbb{C}$.

$H \rightarrow 2n$ eigenvalues

$\left.\begin{array}{l} n \rightarrow \text{in } \mathbb{C}^- \\ n \rightarrow \text{in } \mathbb{C}^+ \end{array}\right\}$ & there will be no eigenvalues on imaginary axis.

If Δ holds, then we define

a function "Ric : $\mathcal{H} \longrightarrow \mathcal{X}$



The set of H matrices which satisfy Δ

The output of Ric operator

$$\mathrm{Ric}(H) = X = X_2 X_1^{-1}.$$

→ Let H belongs to the domain of Riccati operator, and $X = Ric(H)$ (This implies the relation Ⓐ holds).

Then,

(i) $X = X_2 X_1^{-1}$ is symmetric

(ii) X satisfies the ARE:

$$A^T X + XA + XR^{-1}X + Q = 0$$

(iii) $(A + R^{-1}X)$ is Hurwitz (all the eigenvalues are in $\mathbb{C}^-$)

→ Let $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ be an LTI system.

Suppose that H has the following form:

$$H = \begin{bmatrix} A & -BB^T \\ -C^T C & -A^T \end{bmatrix}$$

with (A, B) is stabilizable & (C, A) is detectable. Then H belongs to the domain of Riccati operator.