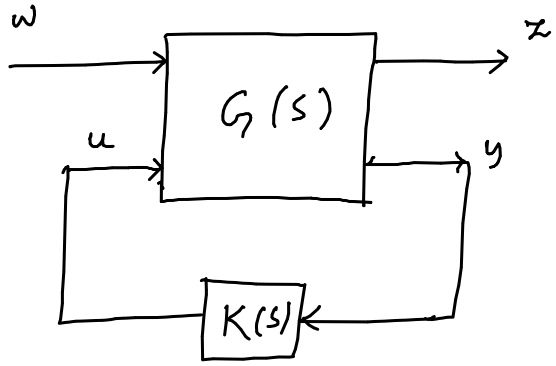
Lecture - 19

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Problems

→ **$\mathcal{H}_2$ - Control Problem**



Problem : Design controller $K(s)$ such that

(i) the closed loop system is internally stable.

(ii) the $\mathcal{H}_2$ - norm of the transfer function (matrix) from w to z is minimum.

→ Why $\mathcal{H}_2$ - norm ?

- Let say we have a signal $u(t)$

  - Instantenous power of $u(t)$ → $(u(t))^2$ $(u^T u)$
  - Total energy of $u(t)$ → $\|u(t)\|_2$

- Average power of $u(t)$:

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (u(t))^2 \, dt$$

(We will say $u(t)$ is a "power signal" if the limit exists)

- R.M.S. value of $u(t)$

$$u_{rms} := \left[\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (u(t))^2 dt \right]^{1/2}$$

- Define autocorrelation of $u(t)$ as follow:

$$R_u(\tau) = E\left[u(t) \, u(t+\tau) \right]$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(t) \, u(t+\tau) \, dt$$

$$R_u(0) = \text{Average power of } u(t)$$

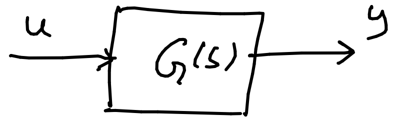
- The power spectral density $S(j\omega)$ of $R_u(\tau)$ is Fourier transform of $R_u(\tau)$

  - $$S_u(j\omega) = \int_{-\infty}^{\infty} R_u(\tau) \, e^{-j\omega\tau} \, d\tau$$

  - $$R_u(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_u(j\omega) \, e^{j\omega\tau} \, d\omega$$

- $R_u(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_u(j\omega)\, d\omega = (u_{rms})^2$

Let say the signals u & y are related as follows:



$$Y(s) = G(s)\, U(s)$$

$y = Gu$ stable & proper

In that case, it can be shown that

$$S_y(j\omega) = |G(j\omega)|^2 S_u(j\omega)$$

$$\left(G^*(j\omega)\, G(j\omega) \right)$$

$$(y_{rms})^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(j\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 S_u(j\omega)\, d\omega$$

If 'u' is white noise then $S_u(j\omega) = 1$

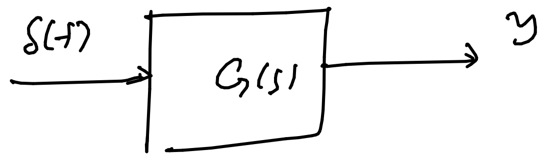
$$(y_{rms})^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2\, d\omega$$

$$= \|G(j\omega)\|_{\mathcal{H}_2}^2$$

* Hence $\mathcal{H}_2$ - norm of a system is equal to the RMS value of the system response to a white noise input.

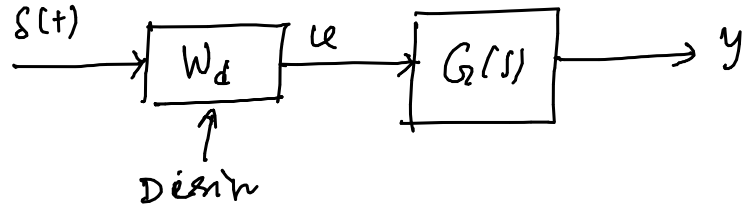
→ The second case

- Recall that $\mathcal{H}_2$ norm of G is equal to the two-norm of the output of the system when the input is an impulse function.



$$\|y\|_2 = \|G\|_{\mathcal{H}_2}$$

- When there is a finite energy signal $u(t)$ which is input to $G(s)$, then we can design $W_d(s)$ (filter) s.t. the output of $W_d(s)$ is $u(t)$ when the input of $W_d(s)$ is impulse function.

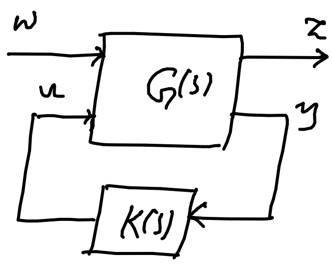


$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \longrightarrow G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

$$C(sI-A)^{-1}B + D$$

Let the system be represented as follows

$$\begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w + D_{22} u \end{cases}$$



z, y } outputs

w, u } inputs

Taking the Laplace transformation:

$$X(s) = (sI-A)^{-1}\left[B_1 W(s) + B_2 U(s)\right]$$

$$Z(s) = \left(C_1(sI-A)^{-1}B_1 + D_{11}\right)W + \left[C_1(sI-A)^{-1}B_2 + D_{12}\right]U$$

$$Y(s) = \left[C_2(sI-A)^{-1}B_1 + D_{21}\right]W + \left[C_2(sI-A)^{-1}B_2 + D_{22}\right]U$$

$$\begin{bmatrix} Z \\ Y \end{bmatrix} = \underbrace{\begin{bmatrix} C_1(sI-A)^{-1}B_1 + D_{11} & C_1(sI-A)^{-1}B_2 + D_{12} \\ C_2(sI-A)^{-1}B_1 + D_{21} & C_2(sI-A)^{-1}B_2 + D_{22} \end{bmatrix}}_{G(s)} \begin{bmatrix} W \\ U \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} Z \\ Y \end{bmatrix} = \left[\begin{array}{c|c} G_{11}(s) & G_{12}(s) \\ \hline G_{21}(s) & G_{22}(s) \end{array}\right] \begin{bmatrix} W \\ U \end{bmatrix}$$

The t.fn from w to z is $H(s)$: (lower LFT)

$$\boxed{H(s) := G_{11}(s) + G_{12}(s)K(s)\left(I - G_{22}(s)K(s)\right)^{-1}G_{21}(s)}$$

Recall that $\mathcal{H}_2$ norm are defined for stable & strictly proper transfer function (matrix).

Assume $D_{11} = 0$ then $G_{11}(s)$ is strictly proper

Does not matter [$D_{22} = 0$ then $G_{22}(s)$ is strictly proper]

Let us assume that $K(s)$ is designed in such a way that it is strictly proper.

Then $\underline{H(s) \text{ is strictly proper.}}$

So $\| H(s) \|_{\mathcal{H}_2}$ is well-defined

→ Suppose G is a system with state space realization : $G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$ satisfying

(i) (A, B_1) is stabilizable and (A, C_1) is detectable.

(ii) (A, B_2) is stabilizable and (A, C_2) is detectable,

(iii) $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

(iv) $D_{21} \begin{bmatrix} B_1^T & D_{21}^T \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

Then the optimal stabilizing controller for the $\mathcal{H}_2$ - synthesis problem is

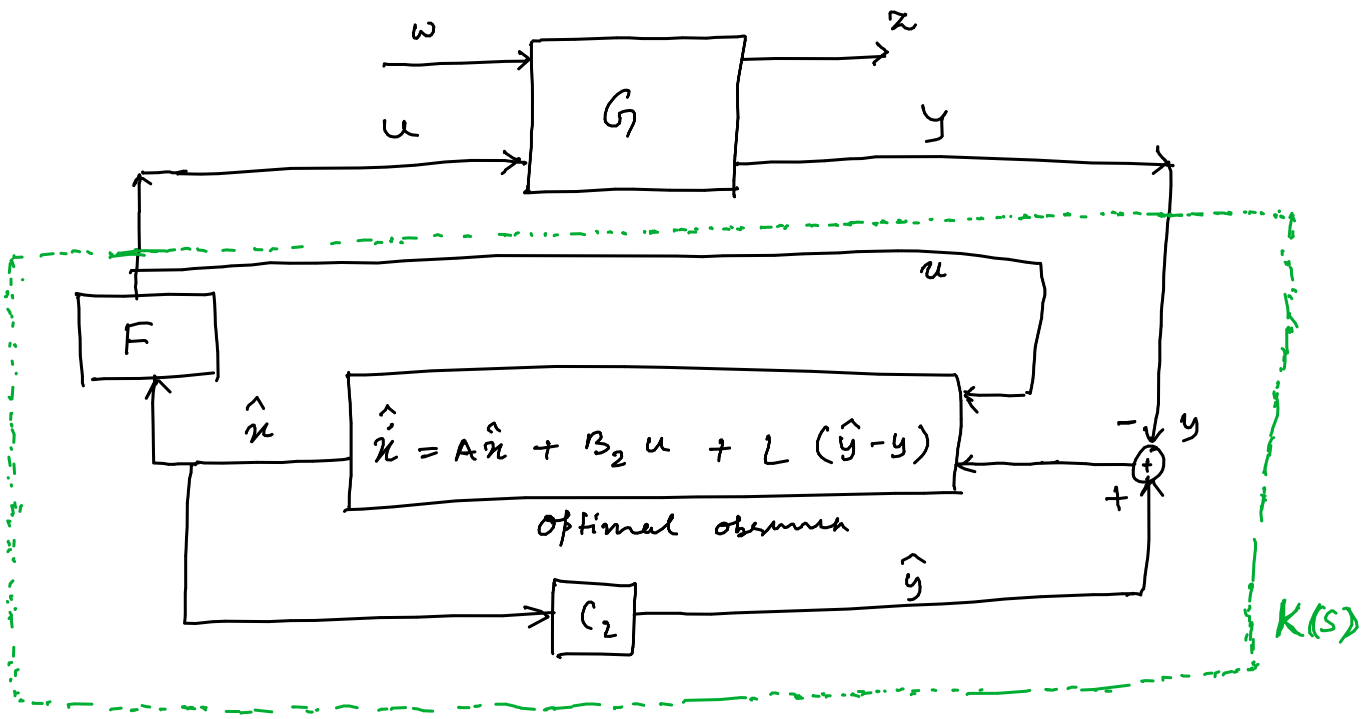
$$K(s) = \left[\begin{array}{c|c} A + B_2 F + L C_2 & -L \\ \hline F & 0 \end{array}\right] \checkmark$$

with $L = -Y C_2^T$ $F = -B_2^T X$ where

$X = Ric(H_2)$ and $Y = Ric(J_2)$ and the associated Hamiltonian matrices are

$$H_2 = \begin{bmatrix} A & -B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{bmatrix} \quad \& \quad J_2 = \begin{bmatrix} A^T & -C_2^T C_2 \\ -B_1 B_1^T & -A \end{bmatrix}.$$

$K(s)$ has S.S.R
$$\begin{cases} \dot{\hat{x}} = (A + B_2 F + L C_2)\hat{x} + (-L)\underset{\substack{\uparrow \\ \text{input}}}{y} \\ \underset{\substack{\uparrow \\ \text{output}}}{u} = F\hat{x} \end{cases}$$



Solve two Riccati equations associated with H & J & then compute the gain matrices F & L.

→ Reasons for the assumptions:

- Assumptions (i) & (ii) ensure that both H & J belong to the domain of Riccati operator.
- Assumption (ii) is necessary to ensure that a stabilizing controller exists.
- Assumptions (iii) & (iv) are made for convenience, for which a simplified controller expression is obtained. (In fact without these assumptions the derivation would be complicated)

- The Assumption (iii) says that the plant output $\underline{C_1 x}$ is orthogonal to the weight on the control effort $D_{12}u$

- The Assumption (iv) says that the system disturbance & measurement noise are orthogonal.

* The $\mathcal{H}_2$ optimal controller we obtained is <u>unique</u>. The <u>optimal cost</u> is

$$\|G_c B_1\|_{\mathcal{H}_2}^2 + \|F G_f\|_{\mathcal{H}_2}^2$$

where

$$G_c = \left[\begin{array}{c|c} A + B_2 F & I \\ \hline C_1 + D_{12} F & 0 \end{array}\right] \qquad G_f = \left[\begin{array}{c|c} A + L C_2 & B_1 + L D_{21} \\ \hline I & 0 \end{array}\right]$$