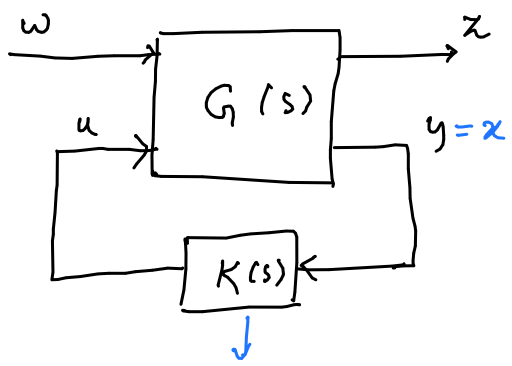
Lecture - 20

→ Static state feedback Control for $\mathcal{H}_2$ - Control Problem
↓
Formulation as an LMI feasibility Problem



$K(s) = F$ is a real matrix

→ Result

Let $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ be a realization

of the transfer function matrix $G(s)$,

Then the matrix A is Hurwitz

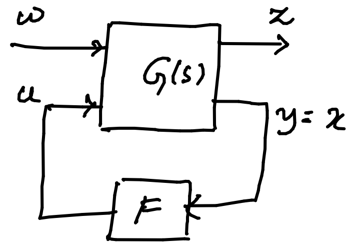
and $\|G(s)\|_{\mathcal{H}_2} < 1$

if and only if there exists a symmetric positive definite matrix $X > 0$ s.t.

$$\mathrm{Tr}\left[CXC^T\right] < 1$$

and $$AX + XA^T + BB^T < 0$$

Let us consider the general state space representation of Fig-1



$$\begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \\ y = x \end{cases}$$

$\boxed{u = Fx}$ static state feedback control

- We design F such that
  (i) feedback configuration is internally stable
  (ii) the t.f.m. $H(s)$ (from w to z) has
  $$\|H(s)\|_{H_\infty} < 1$$

$$\begin{cases} \dot{x} = Ax + B_1 w + B_2 F x \\ \quad = (A + B_2 F)x + B_1 \underline{w} \\ z = C_1 x + D_{12} F x \\ \quad = (C_1 + D_{12} F) x \end{cases}$$

w is input & z is o/p

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Let the t.f. matrix from w to z is $H(s)$.

We use the previous result.

We have to ensure that

$$\left.\begin{array}{l} Tr\left[(C_1 + D_{12}F)X(C_1 + D_{12}F)^T\right] < 1 \\ \text{and} \quad X > 0 \\ (A + B_2F)X + X(A + B_2F)^T + B_1B_1^T < 0 \end{array}\right\} \text{Constraints}$$

Then $(A + B_2F)$ is Hurwitz

$$\|H(s)\|_{\mathcal{H}_2} < 1$$

Constraints are not linear due to the presence of two matrix variables F & X

Introduce a new variable $\underline{\underline{Y = FX}}$

$$\Rightarrow F = YX^{-1}$$

Then the constrains are

(i) $X > 0$

(ii) $Tr\left[C_1XC_1^T + C_1XF^TD_{12}^T + D_{12}FXC_1^T + D_{12}FXF^TD_{12}^T\right] < 1$

by replace $FX = Y$

$$\downarrow$$

$$Tr\left[C_1XC_1^T + C_1Y^TD_{12}^T + D_{12}YC_1^T + D_{12}\underline{\underline{Y}}F^TD_{12}^T\right] < 1$$

problem

iii) $AX + B_2FX + XA^T + XF^TB_2^T + B_1B_1^T < 0$

$\downarrow$

$AX + B_2Y + XA^T + Y^TB_2^T + B_1B_1^T < 0$

linear in X & Y

- A property of <u>Tr</u> function (for symmetric matrices)

If $A > B \Leftrightarrow A - B > 0$

Then $Tr(A) > Tr(B)$ (monotonicity property)

The constraints
$$\begin{cases} X > 0 \\ Tr(PX^{-1}P^T) < 1 \end{cases}$$

can be written in LMI form as follows by introducing a slack variable Z:

$$\begin{cases} \begin{bmatrix} X & P^T \\ P & Z \end{bmatrix} > 0 \rightarrow & Z > 0 \\ & \underbrace{Z}_{A} - \underbrace{PX^{-1}P^T}_{B} > 0 \\ Tr(Z) < 1 & \end{cases}$$

$\downarrow$

Since $A - B > 0$

$Tr(A) > Tr(B)$

i.e. $Tr(Z) > Tr(PX^{-1}P^T)$

$$\mathrm{Tr}\left[(C_1 + D_{12}F) X (C_1 + D_{12}F)^T\right] < 1$$

$$|||$$

$$\mathrm{Tr}\left[(C_1 X + D_{12} Y) X^{-1} (C_1 X + D_{12} Y)^T\right] < 1$$

$$\downarrow$$

$$\mathrm{Tr}\left\{C_1 X C_1^T + C_1 Y^T D_{12}^T + D_{12} Y C_1^T + D_{12} \underbrace{Y X^{-1}}_{F} Y^T D_{12}^T\right] < 1$$

Introduce a slack variable Z, then

$$\left.\begin{array}{l} \begin{bmatrix} X & (C_1 X + D_{12} Y)^T \\ C_1 X + D_{12} Y & Z \end{bmatrix} > 0 \\ \\ \text{and} \qquad \mathrm{Tr}(Z) < 1 \end{array}\right\} \text{LMIs}$$

To compute F that satisfies

(i) $(A + B_2 F)$ is stable (internal stability)

(ii) $\|H(s)\|_{\mathcal{H}_2} < 1$

Can be formulated as follows LMI feasibility problem

Find Y, Z and X s.t.

(i) $X > 0$

(ii)
$$\begin{bmatrix} X & (C_1 X + D_{12} Y)^T \\ C_1 X + D_{12} Y & Z \end{bmatrix} > 0$$

(iii) $Tr(Z) < 1$

(IV) $AX + B_2 Y + XA^T + Y^T B_2^T + B_1 B_1^T < 0$

Use LMI toolbox or SeDuMi } in MATLAB

After solving: Compute $\underline{\underline{F = YX^{-1}}}$.