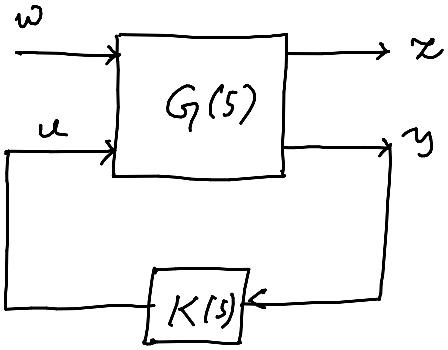
Lecture - 21

$\mathcal{H}_\infty$ - Control Problem



Find controller $K(s)$ such that

- the closed loop system is internally stable
- $\|H_{wz}(s)\|_{\mathcal{H}_\infty}$ is minimum where

  $H_{wz}(s)$ is the t.f. (matrix) from w to z.

- The above problem is difficult to address theoretically & computationally.
- The optimal controller for MIMO case is <u>not unique</u>.

Hence, a suboptimal problem is solved, which is as follows:

- Given $\gamma > 0$, find $K(s)$ (if exists) such that

  (i) the closed loop system is internally stable

  (ii) $\|H_{wz}(s)\|_{\mathcal{H}_\infty} < \gamma$.

↳ this problem is easy to address.

$$u \longrightarrow \boxed{G(s)} \longrightarrow y \qquad\qquad G(s) \in \Omega^{s}_{pr}$$

$$\|G(s)\|_{\mathcal{H}_\infty} := \sup_{\substack{u(t) \in \mathcal{L}_2 \\ \|u(t)\|_2 \neq 0}} \frac{\|y\|_2}{\|u\|_2}$$

$\underline{y} = S\,\underline{d}$

↑ sensitivity

$\mathcal{H}_\infty$ - control problem

↓

"worst-case" optimization problem

Because it minimizes the effect on the output of the system with respect to worst disturbance.

$$\|G(s)\|_{\mathcal{H}_\infty} = \sup_{\omega} \|G(j\omega)\|_2$$ ← induced matrix two-norm.

$$\|AB\|_2 \leq \|A\|_2 \|B\|_2$$

$$\|G(j\omega)H(j\omega)\|_{\mathcal{H}_\infty} \leq \|G(j\omega)\|_{\mathcal{H}_\infty} \|H(j\omega)\|_{\mathcal{H}_\infty}$$ (this property does not hold for $\mathcal{H}_2$-norm)

→ Generalized state space representation of $G(s)$

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \qquad \begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned}$$

Assumptions :

(i) (A, B_1) is stabilizable & (A, C_1) is detectable

(ii) (A, B_2) is stabilizable & (A, C_2) is detectable.

(iii) $D_{12}^T \left[C_1 \;\; D_{12} \right] = \left[0 \;\; I \right]$

(iv) $D_{21} \left[B_1^T \;\; D_{21}^T \right] = \left[0 \;\; I \right]$

With the above assumptions we have the following result :

Let us define two Hamiltonian matrices :

$$H_\infty = \begin{bmatrix} A & \frac{1}{\gamma^2} B_1 B_1^T - B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{bmatrix} \qquad J_\infty = \begin{bmatrix} A^T & \frac{1}{\gamma^2} C_1^T C_1 - C_2^T C_2 \\ -B_1 B_1^T & -A \end{bmatrix}$$

where $\gamma > 0$.

Result

For the generalized state space model, there exists $K(s)$, which internally stabilizes the closed loop system and $\| H_{wz}(s) \|_{\mathcal{H}_\infty} < \gamma$ if and only if

(i) $H_\infty \in dom(Ric)$ and $X_\infty := Ric(H_\infty) \geq 0$

(ii) $J_\infty \in dom(Ric)$ and $Y_\infty := Ric(J_\infty) \geq 0$

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$.

Condition (i)

$$A X_\infty + X_\infty A^T + C_1^T C_1 + \frac{1}{\gamma^2} X_\infty B_1 B_1^T X_\infty - X_\infty B_2 B_2^T X_\infty = 0$$

Sol is X_∞ & Hamiltonial matrix H_∞.

$H_\infty \in dom(Ric)$: H_∞ belongs to the domain of Riccati operator.

condition (ii)

$$A^T Y_\infty + Y_\infty A + B_1 B_1^T + \frac{1}{\gamma^2} Y_\infty C_1^T C_1 Y_\infty - Y_\infty C_2^T C_2 Y_\infty = 0$$

sol Y_∞

cond^n - (iii)

$\rho(A)$: spectral radius of A $\quad A \in \mathbb{R}^{n \times n}$

$$\rho(A) := \max_{1 \le i \le n} |\lambda_i(A)|$$

$$\rho(X_\infty Y_\infty) = \max_{1 \le i \le n} |\lambda_i(X_\infty Y_\infty)| < \gamma^2$$

↳ when the above 3-conditions hold, one such controller:

$$K_{sub}(s) = \left[\begin{array}{c|c} A_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

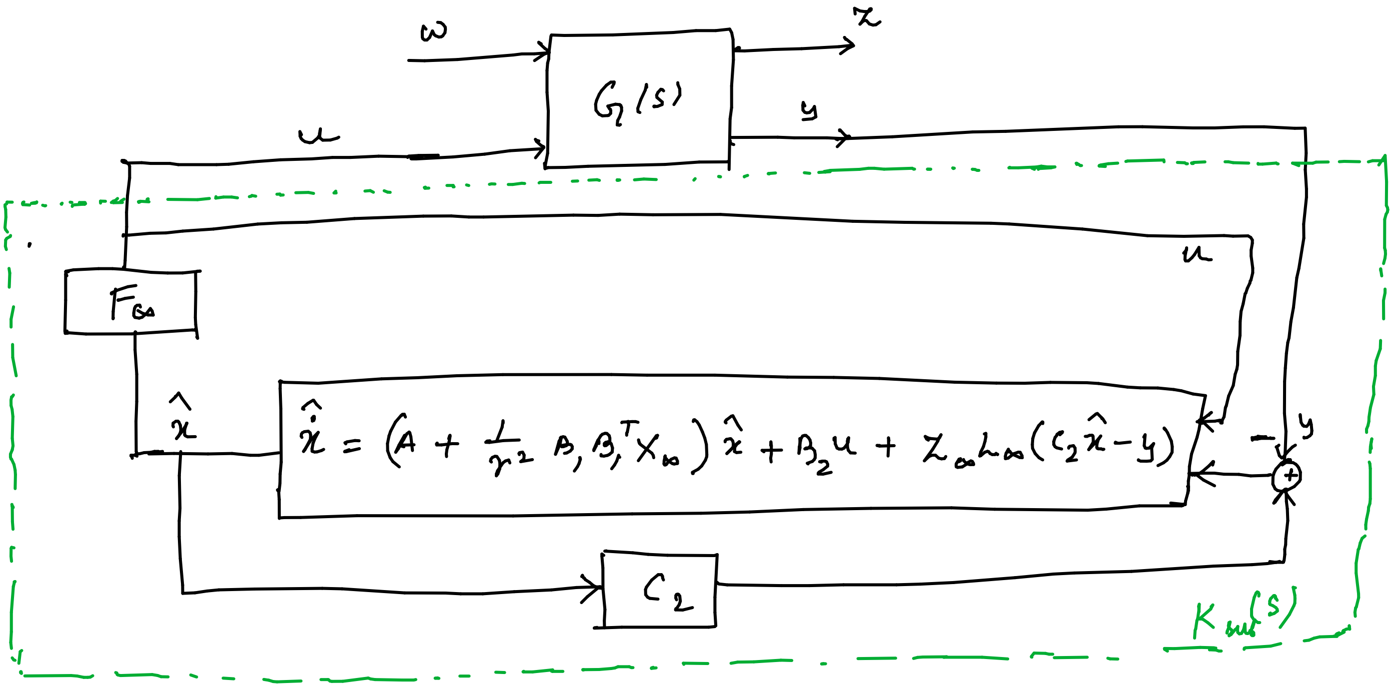
when $A_\infty = A + \frac{1}{\gamma^2} B_1 B_1^T X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2$

$F_\infty = -B_2^T X_\infty \qquad L_\infty = -Y_\infty C_2^T \qquad Z_\infty = \left(I - \frac{1}{\gamma^2} Y_\infty X_\infty\right)^{-1}$

S.S.R of $K_{sub}(s)$ is

$$\dot{\hat{x}} = A_\infty \hat{x} + (-Z_\infty L_\infty) y \leftarrow \text{input}$$

$$\text{output} \rightarrow u = F_\infty \hat{x}$$



↳ separation principle holds

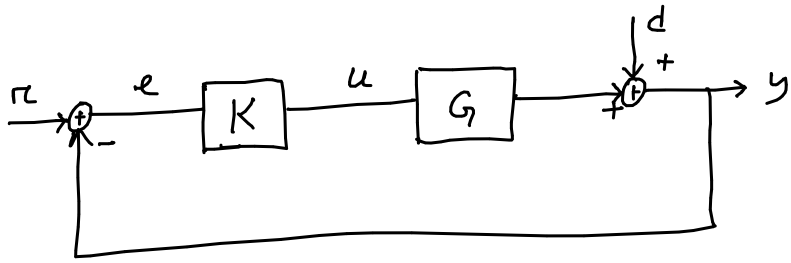
independently design F_∞ & L_∞

$$\frac{1}{\gamma^2} B_1 B_1^T X_\infty \hat{x} \rightarrow B_1 w_{worst}$$

$$\downarrow$$

$$\frac{1}{\gamma^2} B_1^T X_\infty \hat{x}$$

Solve the Riccati eq^ns & obtain $X_\infty \geq 0$, $Y_\infty \geq 0$ } then compute F_∞, L_∞, Z_∞



Let $r=0$

$$y = S d \qquad e = -S d$$

$S \rightarrow$ Sensitivity fnt^n

$$u = K e = -K S . d$$

↳ Introduce weighting functions :

$$S \rightarrow W_e S W_d$$

$$KS \rightarrow W_u K S W_d$$

W_e, W_d & $W_u \in \mathcal{R}\mathcal{H}_\infty$ & these are weighting funct^n

Let say the disturbance signal $d \in \mathcal{L}_2$ & it is scaled appropriately s.t. $\|d\|_2 = 1$

$$\sup_{\|d\|_2=1} \|e\|_2 = \underline{\|W_e S W_d\|_{\mathcal{H}_\infty}}$$

$$\sup_{\|d\|_2=1} \|u\|_2 = \|W_u K S W_d\|_{\mathcal{H}_\infty}$$

For disturbance reduction we have to

$$\text{minimize } \|W_e S W_d\|_{\mathcal{H}_\infty}$$

↑

Solving this problem alone, often increases $\|u\|_2$, which is not desirable.

To address this one define a mixed criterion

$$\left\| \begin{bmatrix} W_e S W_d \\ \beta W_u K S W_d \end{bmatrix} \right\|_{\mathcal{H}_\infty}^2 \rightarrow \sup_{\|d\|_2} \left(\|e\|_2^2 + \beta^2 \|u\|_2^2 \right)$$

$$\begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} W_e S W_d \\ W_u K S W_d \end{bmatrix} d \qquad \left\| \begin{bmatrix} e \\ u \end{bmatrix} \right\|_2$$

Optimizing $\left\| \begin{bmatrix} W_e S W_d \\ \beta W_u K S W_d \end{bmatrix} \right\|_{\mathcal{H}_\infty}$ is called

<u>$\mathcal{H}_\infty$ - mixed sensitivity problem</u>

for using $\mathcal{H}_2$ norm $\rightarrow$ $\mathcal{H}_2$ - mixed sensitivity problem.

---- x ---- x ----