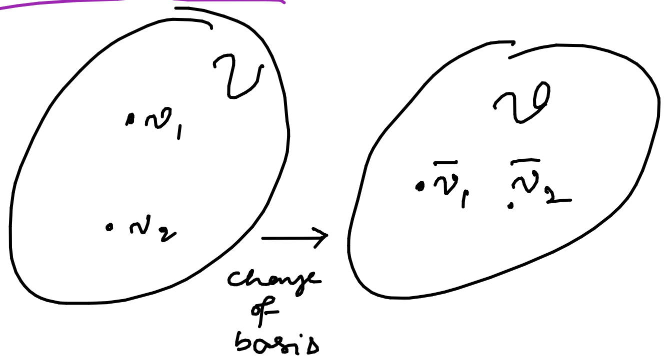
# Lecture - 2 & 3




$$v = \alpha_1 v_1 + \alpha_2 v_2$$ where $\alpha_i$ are scalars.

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$ ← co-ordinate representation vector.

$$v = \beta_1 \bar{v}_1 + \beta_2 \bar{v}_2$$ ← what is the relation between $\alpha$ & $\beta$ ?

$$\alpha_1 v_1 + \alpha_2 v_2 = \beta_1 \bar{v}_1 + \beta_2 \bar{v}_2$$

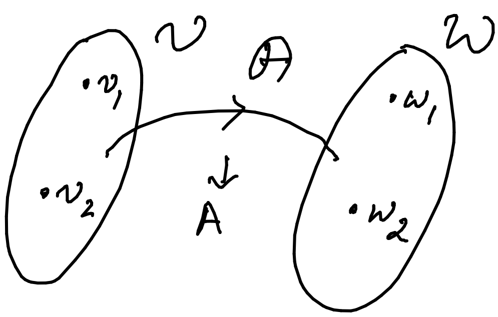
$$\underbrace{\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}}_{V} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \underbrace{\begin{bmatrix} | & | \\ \bar{v}_1 & \bar{v}_2 \\ | & | \end{bmatrix}}_{\bar{V}} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

$V$ & $\bar{V}$ are non-singular matrices.

$$\underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}}_{\beta} = \underbrace{\bar{V}^{-1} V}_{P} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}}_{\alpha}$$

$P$ is also non-singular.

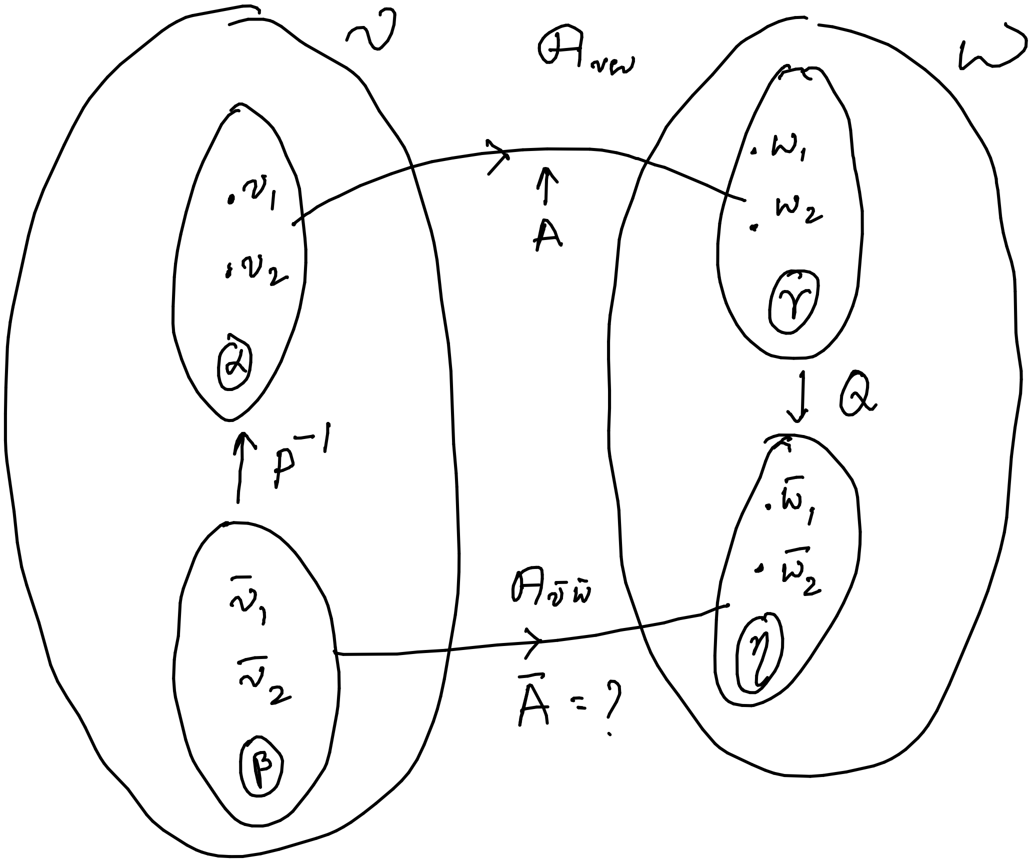
$$\boxed{\beta = P\alpha}$$




$$v = \alpha_1 v_1 + \alpha_2 v_2$$

$$Av = w = \gamma_1 w_1 + \gamma_2 w_2$$

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = A \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$




$\gamma$ & $\eta$ are related

$\eta = Q\gamma$

$\Rightarrow \gamma = Q^{-1}\eta$

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = A \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = AP^{-1}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

$$\gamma = AP^{-1}\beta$$

$$\boxed{\Rightarrow Q^{-1}\eta = AP^{-1}\beta \quad \Rightarrow \eta = \underline{\underline{QAP^{-1}}}\beta}$$

$$\boxed{\eta = \bar{A}\beta} \qquad \boxed{\bar{A} = QAP^{-1}}$$

$$Q = \bar{W}^{-1}W \qquad P = \bar{V}^{-1}V$$

standard basis $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Let $v_i = e_i \in \mathbb{R}^2$

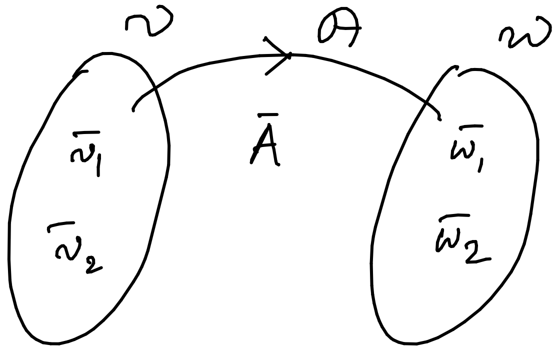
$w_i = e_i$ $\qquad W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = \bar{V}^{-1} V = \bar{V}^{-1} \qquad Q = \bar{W}^{-1} W = \bar{W}^{-1}$$

$$\boxed{\begin{aligned} \bar{A} &= Q A P^{-1} \\ \bar{A} &= \bar{W}^{-1} A \bar{V} \end{aligned}}$$

w.r.t. the original basis which are standard basis.

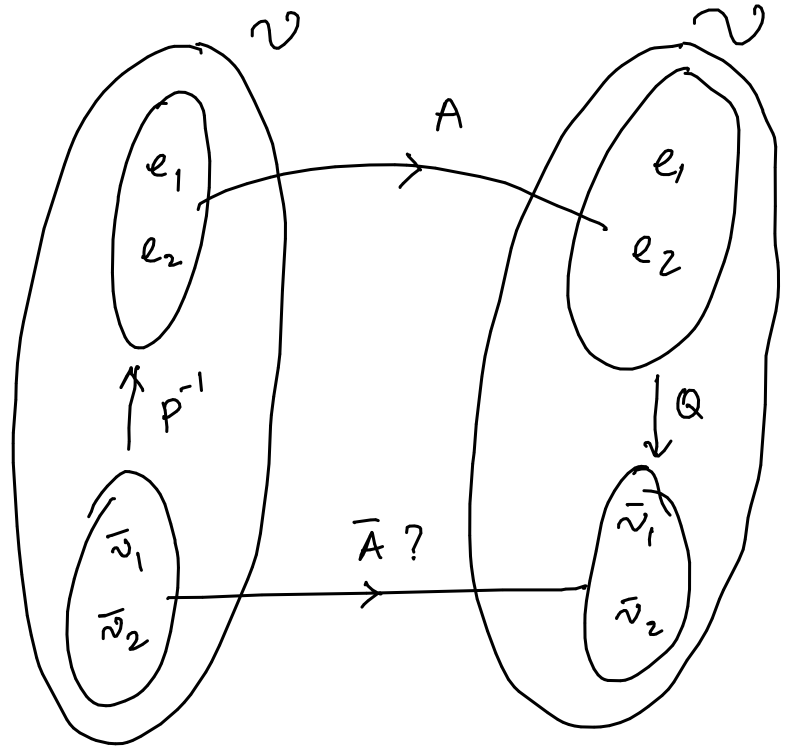



Assume that the given matrix $A$ is w.r.t. the choice of standard basis.

$$\left.\begin{aligned} A\bar{v}_1 &= \tilde{a}_{11} \bar{w}_1 + \tilde{a}_{21} \bar{w}_2 \\ A\bar{v}_2 &= \tilde{a}_{12} \bar{w}_1 + \tilde{a}_{22} \bar{w}_2 \end{aligned}\right\}$$

$$A \underbrace{\begin{bmatrix} \bar{v}_1 & \bar{v}_2 \end{bmatrix}}_{\bar{V}} = \underbrace{\begin{bmatrix} | & | \\ \bar{w}_1 & \bar{w}_2 \\ | & | \end{bmatrix}}_{\bar{W}} \underbrace{\begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{bmatrix}}_{\bar{A}}$$

$$A\bar{V} = \bar{W}\bar{A} \Rightarrow \bar{A} = \bar{W}^{-1} A \bar{V}$$




$v = I$

$$P^{-1} = [\bar{V}^{-1} V]^{-1} = V^{-1}\bar{V} = \bar{V}$$

$$Q = \bar{V}^{-1} V = \bar{V}^{-1}$$

so $\bar{A} = Q A P^{-1}$

$$\boxed{\Rightarrow \bar{A} = \bar{V}^{-1} A \bar{V}}$$

$A$ & $\bar{A}$ are similar matrices.

→ Basis changes are required to obtain a structured matrix which may be useful for several analysis purpose such as (i) solving set of differential eq^ns
(ii) stabilizability
(iii) Detectability

$$[b, \quad Ab \quad A^2 b]$$

$$A \longrightarrow \begin{bmatrix} 0 & 0 & * \\ 1 & 0 & * \\ 0 & 1 & * \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix} \checkmark$$

$$\alpha(s) = |(sI - A)| = s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$$

$$A \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix}$$

$\begin{bmatrix} b & Ab & A^2b \end{bmatrix}$

$Ab = (0)\, b + (1)\, Ab + (0)\, A^2 b$

$A^2 b = (0)\, b + (0)\, Ab + (1)\, A^2 b$

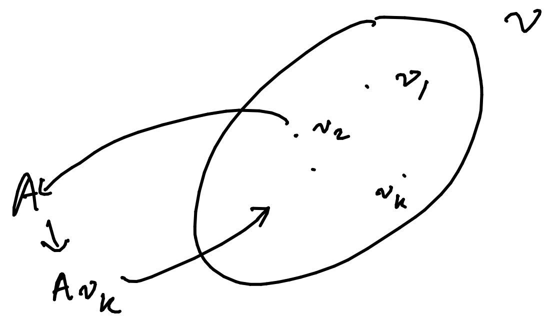
$A^3 b = (*)\, b + (*)\, Ab + (*)\, A^2 b \leftarrow$ Cayley - Hamilton Relat.

The represent of $\bar{A}$ depends on

(1) the choice of basis

(2) the order of the basis

## Invariant Subspace




A - invariant subspace.

→ Let say we are given with square matri.

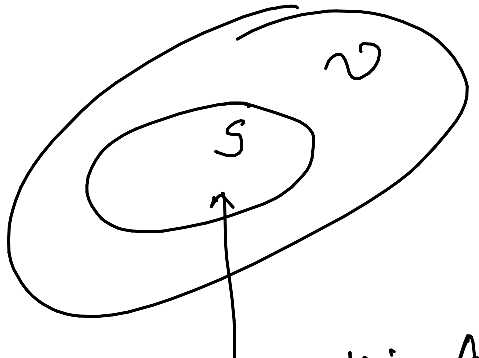
say a vector space $\mathcal{V} \subseteq \mathbb{R}^n$

$A \in \mathbb{R}^{n \times n}$ $\quad v \in \mathcal{V}$

$Av \in \mathcal{V}$

In vector space $\mathcal{V}$, the space $\mathcal{V}$ is the largest A-invariant subspace

$A \cdot 0 = 0 \in \mathcal{V}$ $\quad$ 0 - subspace is the smallest A-invariant subsp.




Is this A-invariant?

$\boxed{b \quad Ab \quad A^2b} \quad v_1 \quad v_2 \quad v_3$

A-invariant subspace

For a given A
What is the matrix represent$^n$ w.r.t. the

basis $\{\underbrace{b, Ab, A^2b}_{A\text{-invariant}}, v_1, v_2, v_3\}$

$$Ab = 0\cdot b + 1\cdot Ab + 0\cdot A^2b + 0\cdot v_1 + 0\cdot v_2 + 0\cdot v_3$$

$$A^2b = 0\cdot b + 0\cdot Ab + 1\cdot A^2b + 0\cdot v_1 + 0\cdot v_2 + 0\cdot v_3$$

$$A^3b = *\cdot b + *\cdot Ab + *\cdot A^2b + 0\cdot v_1 + 0\cdot v_2 + 0\cdot v_3$$

$$Av_1 = * b + * Ab + * A^2b + *\cdot v_1 + *\cdot v_2 + *\cdot v_3$$

$$Av_2 = *\cdot b + * Ab + * A^2b + * v_1 + * v_2 + * v_3$$

$$Av_3 = * b + * Ab + * A^2b + * v_1 + * v_2 + * v_3$$

$$\tilde{A} = \left[\begin{array}{ccc|ccc} 0 & 0 & * & * & * & * \\ 1 & 0 & * & * & * & * \\ 0 & 1 & * & * & * & * \\ \hline 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{array}\right] \qquad \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}$$