Lecture - 4

Norms

vector, matrix & signals

Norm of a vector : It is is function $\|\cdot\|: \mathbb{R}^n \to \mathbb{R}$ & it has following properties: (for $x \in \mathbb{R}^n$)

(i) $\|x\| > 0$ & $\|x\| = 0$ iff $x = 0$

(ii) $\|\alpha x\| = |\alpha| \|x\|$ where $\alpha \in \mathbb{R}$

(iii) $\|x + y\| \leq \|x\| + \|y\|$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Example

$$\|x\|_2 := \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$$

$$\|x\|_1 := |x_1| + |x_2| + \cdots + |x_n|$$

$$\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|$$

$$\|x\|_p := \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p \right)^{1/p}$$

Signal norms

$$\underbrace{\text{signals } u(t)}$$

scalar valued signals

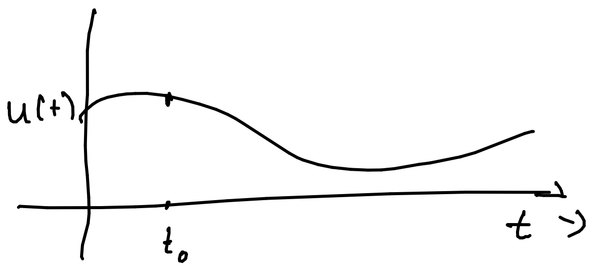
$u : (-\infty, \infty) \longrightarrow \mathbb{R}$

vector valued signal

$u : (-\infty, \infty) \longrightarrow \mathbb{R}^n$

$t \in (-\infty, \infty)$

$u(t) \longrightarrow u : (-\infty, \infty) \longrightarrow \mathbb{R}$

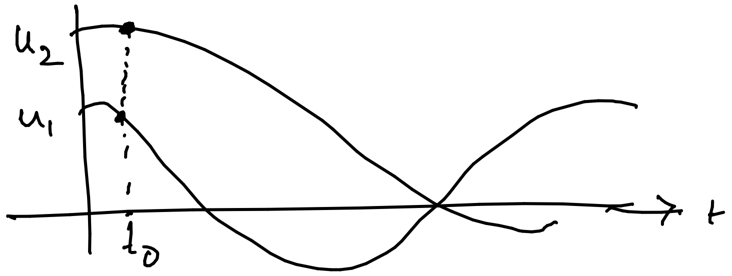


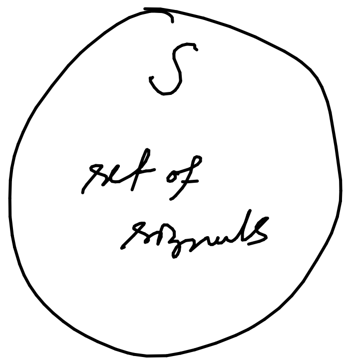
$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}$$

$t \in (-\infty, \infty)$

$(-\infty, \infty) \longrightarrow \mathbb{R}^n$

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$





Definition: Norm of a signal is funct^n

$\|\cdot\| : S \to \mathbb{R}$ which satisfies the following properties:

(i) $\|u(t)\| > 0$ if $u(t) \neq 0$

& $\|u(t)\| = 0$ iff $u(t) \equiv 0$ for all t

(ii) $\|\alpha u(t)\| = |\alpha| \, \|u(t)\|$ $\quad \alpha \in \mathbb{R}$

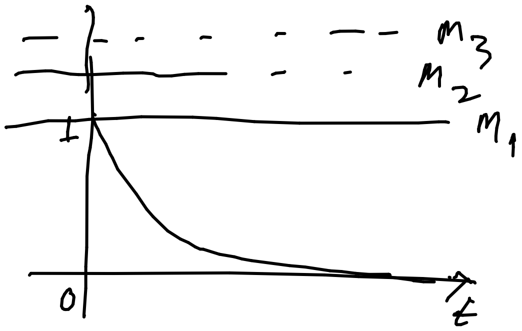
(iii) $\|u(t) + v(t)\| \leq \|u(t)\| + \|v(t)\|$ $\quad \forall t$

1-norm of a signals (scalar valued) $t \in [0, \infty)$

$$\|u(t)\|_1 := \int_{-\infty}^{\infty} |u(t)| \, dt$$

$$:= \int_{0}^{\infty} |u(t)| \, dt$$

$$\|u(t)\|_2 := \left[\int_{-\infty}^{\infty} (u(t))^2 \right]^{1/2}$$



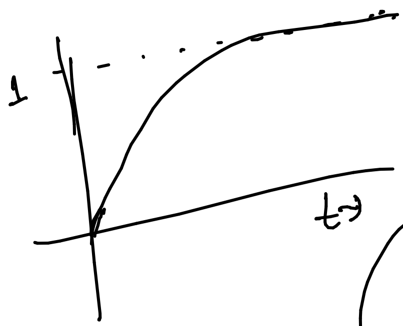
$$\|u(t)\|_\infty := \sup_{t \in (-\infty, \infty)} |u(t)|$$

$t \in (0, \infty) \to \max = ?$ ✗

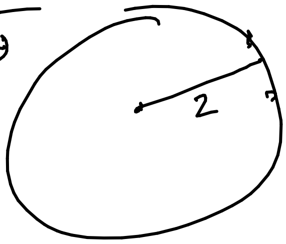
$$\sup_t u(t) := \min \{ M : M \geq u(t), \forall t \}$$

sup = 1

M is a scalar.



$$\min \{ M_1, M_2, M_3 \} = M_1$$



$t \in [0, \infty) = \sup = \max$
$= 1$

→ vector valued funct? :

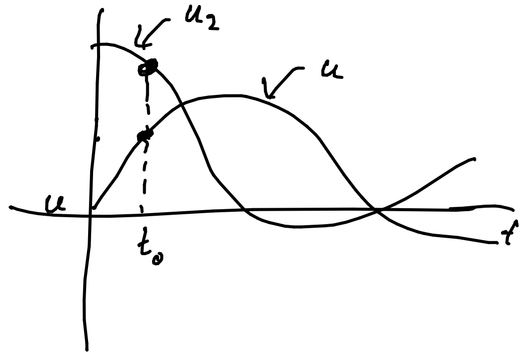
$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}$$

$$\|u(t)\|_1 := \int_{-\infty}^{\infty} \underbrace{\sum_{i=1}^{n} |u_i(t)|}_{\text{Scala at each}} dt \qquad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$t = t_0 \qquad u(t_0) = \begin{bmatrix} u_1(t_0) \\ u_2(t_0) \end{bmatrix} \text{ to}$$

$$\downarrow$$

$$x$$



$$\|u(t)\|_2 := \left[\int_{-\infty}^{\infty} u(t)^T u(t) \, dt \right]^{1/2}$$

$$x^T x = x_1^2 + x^2 + x$$

$$\|u(t)\|_\infty := \sup_{t \in (-\infty, \infty)} \left(\max_{1 \le i \le n} |u_i(t)| \right)$$

→ L_1 space : The space of all signals $u(t)$ which satisfies $\|u(t)\|_1 < \infty$.

→ L_2 space : The space all signals $u(t)$ which satisfies $\|u(t)\|_2 < \infty$

→ L_∞ space : The space of all signals $u(t)$ which satisfied $\|u(t)\|_\infty < \infty$.

Norms of matrices

Definitions: A matrix norm is a function

$$\|\cdot\|_M : \mathbb{R}^{n\times m} \longrightarrow \mathbb{R}$$ such that it

has following properties:

(i) $\|A\|_m > 0$ if $A \neq 0$

(ii) $\|A\|_m = 0$ if $A = 0$

(iii) $\|\alpha A\|_m = |\alpha| \, \|A\|_m$ $\quad \alpha \in \mathbb{R}$

(iv) $\|A+B\|_m \leq \|A\|_m + \|B\|_M$

(v) $\|AB\|_m \leq \|A\|_m \|B\|_M$

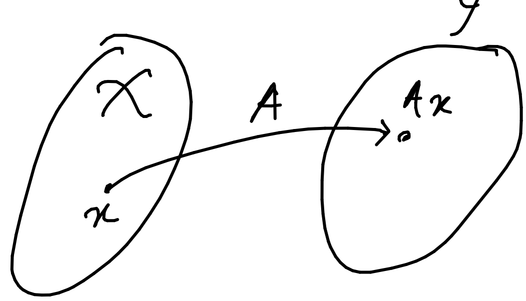
→ Example $\|A\|_M := \left[\sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|^2 \right]^{1/2}$ ← Frobenious norm of a matrix

Induced norm :

$\|\cdot\|_v$ $\quad \|\cdot\|_m$ ← norm for matrix

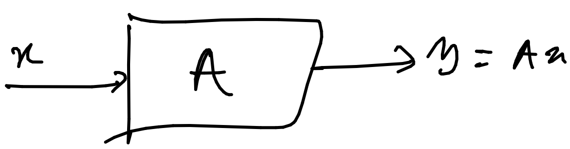
↑ norm for vectors

$$\|A\|_m := \max_x \frac{\|Ax\|_v}{\|x\|_v}$$



operator norm or induced matrix norm

$$\|A\|_2 = \max_x \frac{\|Ax\|_2}{\|x\|_2}$$



For matrix 2 - norm (spectral norm)

$$\|A\|_2 = \sqrt{\underset{\uparrow}{\lambda_{max}}(A^T A)} \quad \text{or} \quad \sqrt{\lambda_{max}(AA^T)}$$

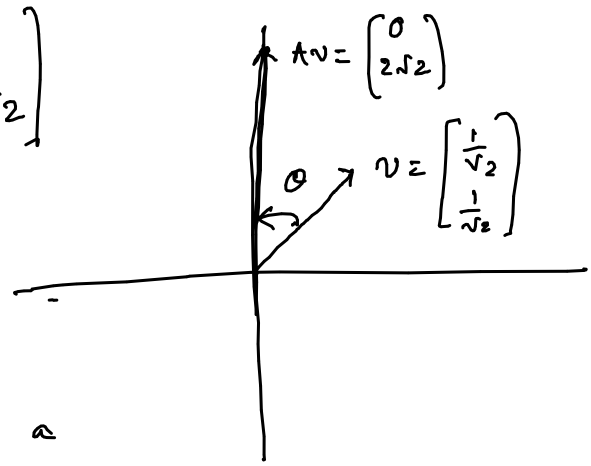
maximum eigenvalue

$$\|A\|_1 := \max_{1 \le j \le m} \left(\sum_{i=1}^{n} |a_{ij}| \right)$$

$$\|A\|_\infty := \max_{1 \le i \le n} \left(\sum_{j=1}^{m} |a_{ij}| \right)$$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \qquad Av = \begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix}$$

$$v = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$



When we operate A on a vector, we do following two operations.

(i) rotation

(ii) magnification

$A = ?$

→ Orthogonal matrix

Orthonormal vectors : $\left.\begin{matrix} x \\ y \end{matrix}\right\} \to x^T y = 0$ ← orthogonal to each other

$$\begin{cases} x^T x = c_x \\ y^T y = c_y \end{cases} \qquad \left.\begin{matrix} x^T x = 1 \\ \& \ y^T y = 1 \end{matrix}\right\} \text{orthonormal}$$

Let say $\{q_1, q_2\}$ be two orthonormal vectors.

$$Q := \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix}$$

$$Q^T Q = \begin{pmatrix} - & q_1^T & - \\ - & q_2^T & - \end{pmatrix} \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} = \begin{bmatrix} q_1^T q_1 & q_1^T q_2 \\ q_2^T q_1 & q_2^T q_2 \end{bmatrix}$$

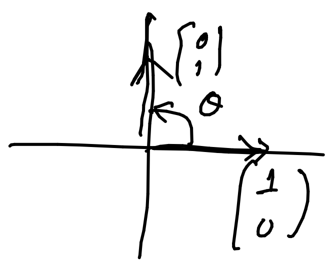
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We will say a matrix Q is orthogonal if it satisfies $\boxed{Q^T Q = Q Q^T = I}$

$$Q^{-1} = Q^T$$

→ $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \to \underset{v}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$$Rv = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Rotaton.

General form of a rotataton matrix $R \in \mathbb{R}^{2\times 2}$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

← will rotate a vector v by angle θ° in anticlockwise.

↳ $R R^T = R^T R = I$

rotations are orthogonal matrices.

$$R = \begin{bmatrix}
1 & & & & & & & & & & \\
 & \ddots & & & & & & & & & \\
 & & 1 & & & & & & & & \\
 & & & c & \cdots & \cdots & & -s & & & \\
 & & & & 1 & & & & & & \\
 & & & & & \ddots & & & & & \\
 & & & & & & 1 & & & & \\
 & & -s & & & & & c & & & \\
 & & & & & & & & 1 & & \\
 & & & & & & & & & \ddots & \\
 & & & & & & & & & & 1
\end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow i^{th} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}$$

↑ j^{th} col.

$c \rightarrow \cos\theta$

$s \rightarrow \sin\theta$

ij^{th} plane rotaton