Lecture - 5

Any matrix that preserving length of vectors but give rotation ?

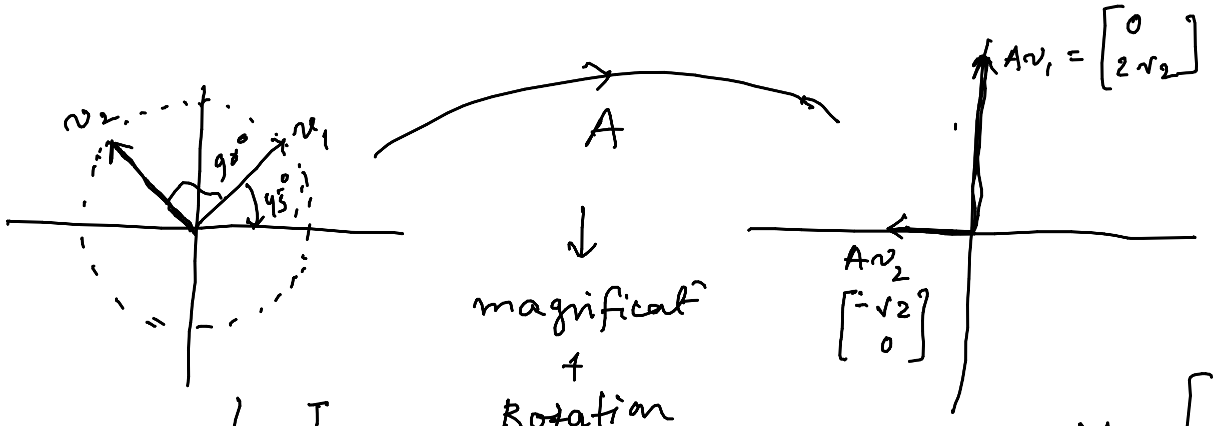
$x \rightarrow$ vector

Qx where Q is orthogonal

$$\underset{x^Tx}{\|x\|_2} \xrightarrow{Q} \|\underbrace{Qx}_{y}\|_2 = \|x\|_2 \qquad Q^TQ = I$$

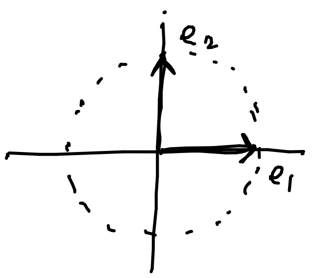
$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad v_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad Av_1 = \begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix} \qquad Av_2 = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}$$



$\downarrow V^T$

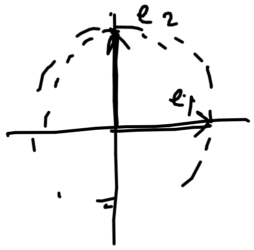
$$\underline{V} = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}$$



$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \text{when } \theta = 45^\circ$$

V^T

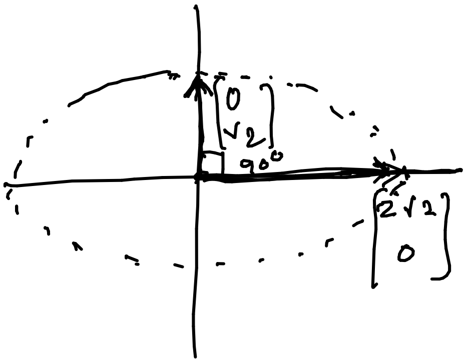
$$V^TV = I$$



$$\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} e_1 & e_2 \end{bmatrix}$$

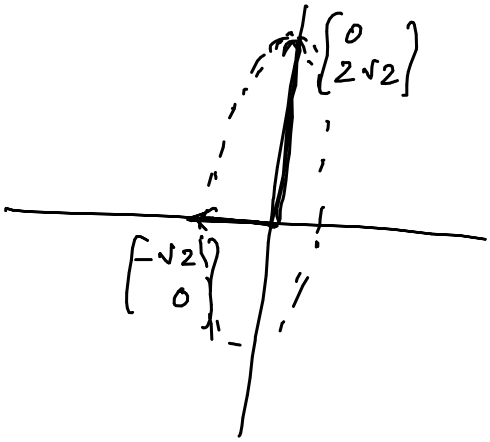
$$= \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

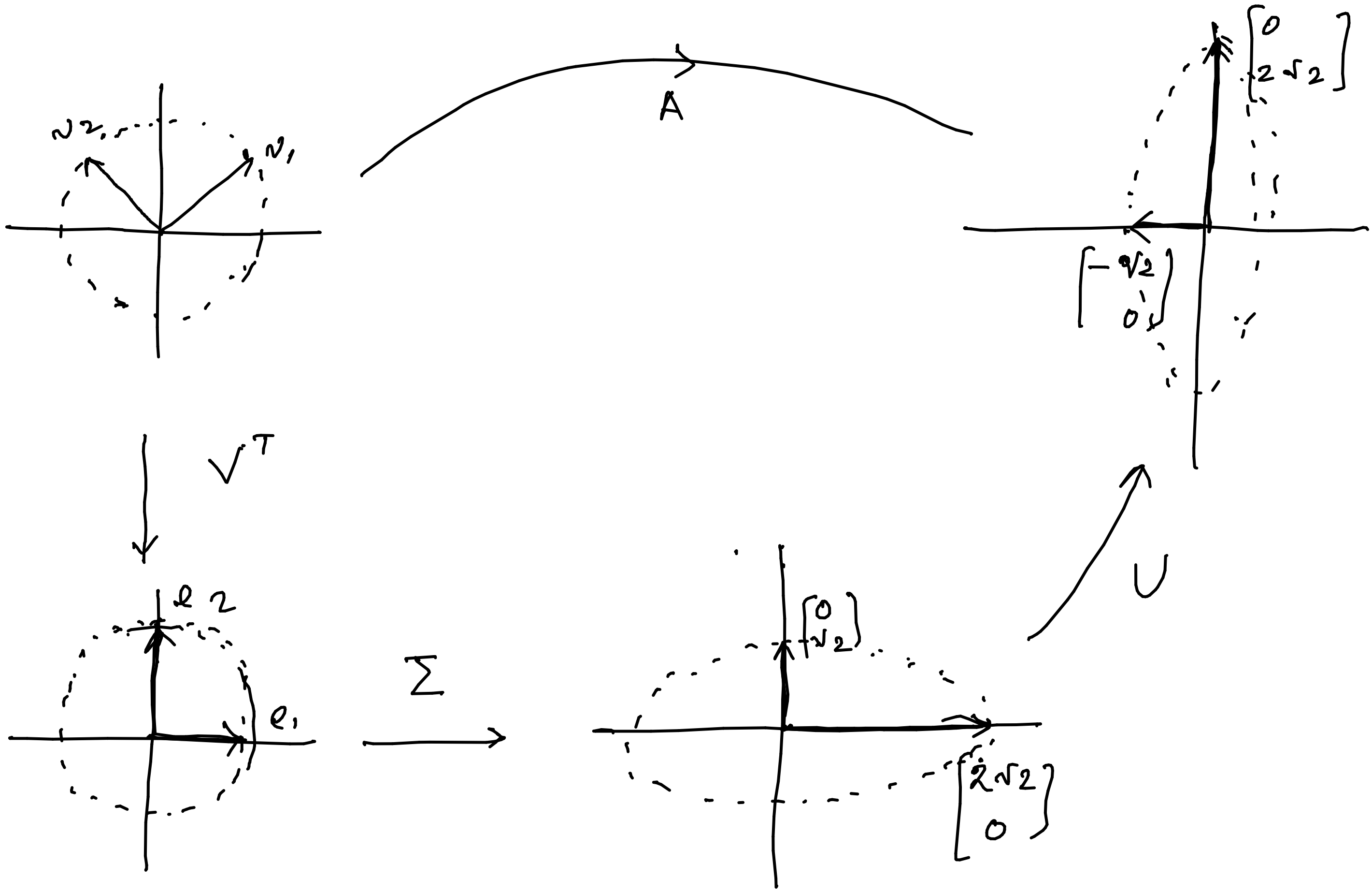
$\downarrow \Sigma$



needs 90° rot^n in anticlockw.

$$U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$





$$\boxed{A = U \Sigma V^T}$$ ⟵ Singular Value Decomposition

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Singular Value Decomposition (SVD)

Let $A \in \mathbb{R}^{n \times m}$ be a non-zero matrix with rank $= r$. The A can be expressed as

$$A = U \Sigma V^T$$

where $U \in \mathbb{R}^{n \times n}$ & $V \in \mathbb{R}^{m \times m}$ are orthogonal matrices & $\Sigma \in \mathbb{R}^{n \times m}$ is a non-square matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & 0 & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & 0 & & & 0 & \\ & & & & & 0 \end{bmatrix}$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

$$\sigma_i = \begin{cases} \text{Singular values of } A \\ \quad\quad \text{or} \\ \text{Principal values} \\ \text{Principal gains} \end{cases}$$

The orthonormal vectors $\rightarrow$ principal directions

$u_i \rightarrow$ right singular vector

$v_i \rightarrow$ left singular vector

$$A = U \Sigma V^T$$

$$\Rightarrow AV = U\Sigma$$

$$V = [v_1 \; v_2 \; \cdots \; v_m] \qquad U = [u_1 \; u_2 \; \cdots \; u_n]$$

$$A v_1 = \sigma_1 u_1$$

$$A v_2 = \sigma_2 u_2$$

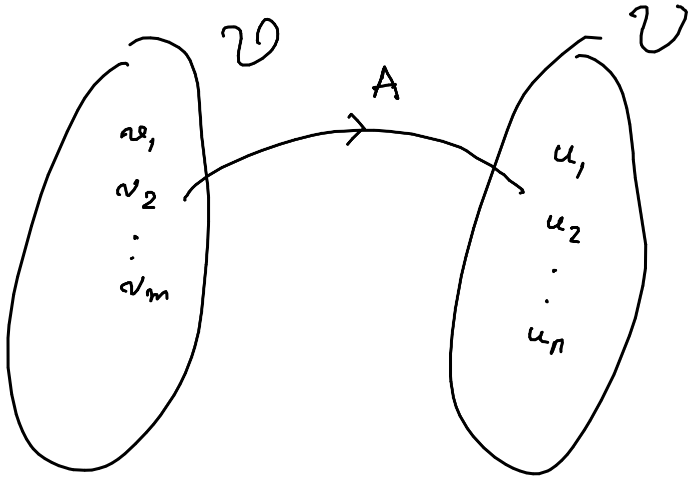
$$\vdots$$

$$A v_r = \sigma_r u_r$$

$$A v_{r+1} = 0$$

$$\vdots$$

$$A v_m = 0$$



v_1 is input for A

$\sigma_1 u_1$ is output " A

Computation of SVD:

$$A = U \Sigma V^T$$

$$A^T A = V \Sigma^T U^T U \Sigma V^T \qquad V^{-1} A V = D_\lambda$$

$$A^T A = V \Sigma^2 V^T$$

$$\Rightarrow V^T \underbrace{(A^T A)}_{\text{symmetric}} V = \Sigma^2$$

← The singular values σ_i of A are the square root of eigenvalues of $A^T A$

$A \in \mathbb{R}^{n \times m}$

- By doing eigenvalue decomposition of $A^T A$ we can compute V & Σ.

$$A \longrightarrow V, \; D_\Lambda$$

$$D_\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$V^{-1} A \underline{V} = \underline{D_\Lambda}$$

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

$$A \longrightarrow A^T A \xrightarrow[\text{Do eigenvalue decomposition}]{} V^T (A^T A) V = \Sigma^2$$

$$AA^T = U \Sigma V^T V \Sigma^T U^T$$

$$= U \Sigma^2 U^T$$

$$\Rightarrow \underline{U^T} (AA^T) \underline{U} = \underline{\Sigma}^2 \longleftarrow$$ this will help to compute U.

$$\boxed{\sigma_i = \sqrt{\lambda_i (A^T A)}}$$

$\lambda_i(\cdot)$: eigenvalues

$$\|A\|_2 := \max_x \frac{\|Ax\|_2}{\|x\|_2} = \sigma_1$$ (the largest singular value of A)

$$\sigma_1 = \sqrt{\lambda_1 (A^T A)}$$

→ Some Results for Linear System Theory :

Let say a system's dynamic behavior is represented by

$$(*) \quad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Following statements are equivalent :

(i) (A, B) is controllable. $\quad A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$

(ii) The controllability matrix

$$\mathcal{C} := \left[B \;\; AB \;\; A^2B \; \cdots \; A^{n-1}B \right]$$

has full <u>row rank</u>.

(iii) The matrix $\left[A - sI \quad B \right]$ has full row rank for all $s \in \mathbb{C}$.

PBH rank test

(iv) Let λ be an eigenvalue of A & the corresponding left eigenvector is w^T ; i.e

$$w^T A = \lambda w^T$$

Then $\quad w^T B \neq 0$

The columns of B are not orthogonal to the left eigenvectors of A.

(v) The eigenvalues of $(A+BF)$ can be freely assigned in a complex plane by a suitable choice of gain matrix 'F'.

(vi) The matrix ← controllability Gramian

$$W_c(t) := \int_0^t e^{A\tau} BB^T e^{A^T\tau} \, d\tau$$

is symmetric positive definite for all $t > 0$.

Observability

Following statements are equivalent:

(i) The pair (A, C) is observable. $C \in \mathbb{R}^{p \times n}$

(ii) The observability matrix

$$\mathcal{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$ is full column rank

(iii) The matrix $\begin{bmatrix} A - sI \\ C \end{bmatrix}$ has full column rank for all $s \in \mathbb{C}$.

(iv) Let λ be an eigenvalue of A & its corresponding right eigenvector be v, i.e

$$Av = \lambda v$$

Then $Cv \neq 0$

the rows of C are not orthogonal to the right eigenvectors of A.

(v) The eigenvalues of $(A+LC)$ can be freely assigned in $\mathbb{C}$ by appropriate choice of matrix L.

(vi) The matrix

Observability Gramian matrix

$$W_o(t) := \int_0^t e^{A^T\tau} C^T C e^{A\tau} \, d\tau$$

is symmetric positive definite $\forall\, t > 0$.