Lecture - 6

Stabilizability & Detectability

→ If a system is controllable then $\mathcal{C}$ is full rank

→ for single input (A, b)

$[b \ Ab \ A^2b \cdots A^{n-1}b]$ as basis → $\text{eig}(A+BF)$ can be assigned arbitrarily

$[b \ Ab \ A^2b \ v_1 \ v_2]$

choose as basis → $\tilde{A} = \begin{bmatrix} A_c & A_{c\bar{c}} \\ 0 & A_{\bar{c}} \end{bmatrix}$ $\quad \tilde{b} = \begin{bmatrix} b_c \\ 0 \end{bmatrix}$

$T^{-1}AT \qquad T^{-1}b$

$f = [f_1 \ f_2]$

$\tilde{b}f = \begin{bmatrix} \times & \times \\ 0 \end{bmatrix}$

$(\tilde{A} + \tilde{b}f)$ → $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ → $\begin{bmatrix} * & * \\ 0 & A_{\bar{c}} \end{bmatrix}$

$A_{\bar{c}}$

The eigenvalues of $A_{\bar{c}}$ are uncontrollable modes.

$\lambda_i((A + bf))$ at arbitrary locat^n using f

The eigenvalues of $A_{\bar{c}}$ are in open left half of $\mathbb{C}$.

- A system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ is "stabilizable"

if all the uncontrollable modes are stable.

$\Updownarrow$

the eigen values of $A_{\bar{c}}$ are stable.

→ Let λ_i be an eigenvalue of matrix A, we will refer λ_i as ith mode of the system.

- A mode λ_i is controllable if

$w_i^T B \neq 0$, where $w_i^T A = \lambda_i w_i^T$

- A mode λ_i is observable ~~iff~~ if

$C v_i \neq 0$ where $A v_i = \lambda_i v_i$

→ Following statements are equivalent:

(i) The pair (A, B) is stabilizable.

(ii) The matrix $[A - sI \quad B]$ has full row rank for all $Re(s) \geq 0$ when $s \in \mathbb{C}$.

(iii) There exists a matrix F s.t. $(A + BF)$ is stable.

Observability

→ (A, C) is observable

$\lambda_i(A + LC)$ can be assigned arbitrarily.

If a system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ is not observable

↓

$$\tilde{A} = T^{-1}AT = \left[\begin{array}{c|c} A_o & 0 \\ \hline A_{o\bar{o}} & A_{\bar{o}} \end{array}\right] \qquad \tilde{C} = [C_o : 0]$$

the eigenvalue of $A_{\bar{o}}$ are unobservable

↑

unobservable modes

- Detectable : A system is "detectable" if all the unobservable modes are stable.

→ Following statements are equivalent.

(i) the pair (A, C) is detectable

(ii) the matrix $\begin{bmatrix} A - sI \\ C \end{bmatrix}$ has full column rank for all $Re(s) \geq 0$ when $s \in \mathbb{C}$

(iii) there exists a matrix L s.t. the eigenvalues of $(A + LC)$ are stable.

For a given LTI system, let $G(s)$ be its transfer funct^n.

$$G(s) = \frac{s-1}{s^2-3s+2}$$

set 1 $\left\{ A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad c = [1 \ \ -1] \quad d = 0 \right.$ ✓

controllable but not observable

set 2 $\left\{ A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad c = [1 \ \ 0] \quad d = 0 \right.$

controllable but not observable

set 3 $\left\{ A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad c = [1 \ \ -1] \quad d = 0 \right.$

Observable but not controllable

For a given t.f $G(s) \rightarrow$ a state space realization

$$G(s) = \frac{s-1}{s^2-3s+2} = \frac{(s-1)}{(s-1)(s-2)} = \frac{1}{s-2}$$

$$\frac{1}{s-1} \longrightarrow \frac{(s-1)(s-3)}{(s-1)(s-3)(s-2)}$$ $\Leftarrow$ Increase the order of the state

$\hookrightarrow$ Minimal Realization of a transferfunct^n?

$\Updownarrow$

the system is both controllable & observable.

Minimal realizat'

$$G(s) \rightarrow \begin{cases} A = 2 & b = 1 \quad c = 1 \\ \dot{x} = 2x + u & y = x \end{cases}$$

→ If the matrix A has smallest possible dimension, then we will say the state space realization is minimal.

↓

A state space realization of G(s) is "minimal" iff (A, B) is controllable & (A, C) is observable

↓

All modes are both controllable & observable.

$$G(s) = \frac{b(s)}{a(s)}$$ where b(s) & a(s) are polynomials.

↓

poles — roots of a(s)

zeros — roots of b(s)

S.S.R → A, b, c, d

$$G(s) = c(sI - A)^{-1} b + d \equiv \frac{b(s)}{a(s)}$$

→ $A \in \mathbb{R}^{n \times n}$ $B \in \mathbb{R}^{n \times m}$ $C \in \mathbb{R}^{p \times n}$ $D = 0$

$$G(s) = C(sI - A)^{-1} B$$

$$G(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) & \cdots & g_{1m}(s) \\ g_{21}(s) & g_{22}(s) & \cdots & g_{2m}(s) \\ \vdots & & & \\ g_{p1}(s) & g_{p2}(s) & \cdots & g_{pm}(s) \end{bmatrix}$$

transfer functn matrix

$$g_{ij}(s) = \frac{b_{ij}(s)}{a_{ij}(s)} \begin{array}{l} \leftarrow \text{polynomials} \\ \leftarrow \text{polynomials} \end{array}$$

Where are the poles & zeros of $G(s)$?