Lecture - 7

- **Poles & Zeros of transfer function matrix:**

- Poles of a given transfer funct^n matrix $G(s)$:

Let us define

$\phi(s) :=$ The least common multiple (LCM) of all denominators of all non-identically zero minors of all order of $G(s)$.

Poles: → The roots of the polynomial $\phi(s)$ are the poles of $G(s)$.

lcm (is the least degree polynomial that is multiple of all polynomials)

Let
$$G(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix}$$

→ Step - 1

Compute the minor of all order

Minors of order 1:

$$\frac{s-1}{(s+1)(s+2)} \quad \frac{s}{(s+1)(s+2)}, \quad \frac{-6}{(s+1)(s+2)}, \quad \frac{s-2}{(s+1)(s+2)}$$

Minor of order 2

$$\frac{s^2+3s+2}{(s+1)^2(s+2)^2} = \frac{1}{(s+1)(s+2)}$$ (Do all cancellations)

* [Cancel the common factors between numerator & denominator.

- Step - 2

Determine the LCM of all denominators of the above minors

$$\phi(s) = (s+1)(s+2)$$

The roots of $\phi(s)$ are the poles of $G(s)$

i.e. $-1, -2$.

- Zeros of $G(s)$

Normal rank of a polynomial matrix:

$$P(s) = \begin{bmatrix} p_{11}(s) & p_{12}(s) & p_{13}(s) \\ p_{21}(s) & p_{22}(s) & p_{23}(s) \end{bmatrix}$$ $p_{ij}(s)$ are polynomials

* It is the maximum dimension of a square submatrix of $P(s)$ s.t. its determinant is not identically zero.

non-zero polynomial

A rational polynomial matrix $G(s)$ can be
(transfer function matrix)

written in the form : $\boxed{G(s) = \frac{1}{d(s)} N(s)}$

where $N(s)$ is a polynomial matrix
$d(s)$ is a polynomial

$$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{1}{s+1} \\ 0 & \frac{1}{(s+1)^2} \end{bmatrix} \rightarrow \underbrace{\frac{1}{(s+1)^2}}_{d(s)} \underbrace{\begin{bmatrix} (s-1)(s+1) & s+1 \\ 0 & 1 \end{bmatrix}}_{N(s)}$$

- The <u>normal rank of $G(s)$</u> is equal to the normal rank of $N(s)$.

- Let we have two polynomials $p(s)$ & $q(s)$.

<u>Common divisor of $p(s)$ & $q(s)$</u> : The polynomial $g(s)$ is a common divisor of $p(s)$ & $q(s)$ if

$$p(s) = g(s)\,\tilde{p}(s)$$
$$q(s) = g(s)\,\tilde{q}(s)$$

where $\tilde{p}(s)$ & $\tilde{q}(s)$ are polynomials.

- Greatest Common Divisor of $p(s)$ & $q(s)$:

The highest degree common divisor $g^*(s)$ of $p(s)$ & $q(s)$ is GCD of $p(s)$ & $q(s)$.

or

$g^*(s)$ is GCD of $p(s)$ & $q(s)$ if any common divisor $g(s)$ of $p(s)$ & $q(s)$ is divisor of $g^*(s)$

i.e.

$$g^*(s) = g(s)\, m(s)$$

- Coprime polynomials :

Two polynomials $p(s)$ & $q(s)$ are co-prime if the gcd of $p(s)$ & $q(s)$ is a non-zero real number.

Zeros of $G(s)$

Let us define a polynomial

$\psi(s)$: the gcd of all numerators of all order-r minors of $G(s)$ (where r is the normal rank of $G(s)$) provided that these minors have been adjusted in a such a way so as to have the pole polynomial $\phi(s)$ as their denominator.

5

The roots of $\psi(s)$ are the zeros of $G(s)$.
(transmission zeros)

→ Let
$$G(s) = \begin{bmatrix} \frac{s-1}{s+2} & \frac{4}{s+2} \\ \frac{4.5}{s+2} & \frac{2(s-1)}{s+2} \end{bmatrix}$$

Step - 1

Compute the pole polynomial $\phi(s)$

$$\phi(s) = (s+2)$$

Step - 2

Compute the normal rank of $G(s)$. Then compute the minor of $G(s)$ having order equal to the normal rank of $G(s)$.

The normal rank $G(s) \rightarrow 2$

Minor of order $2 \rightarrow \frac{2(s-4)}{(s+2)}$

Step - 3

Adjust the all order minors of $G(s)$ so that $\phi(s)$ will appear in the denominator.

For this eg. $\phi(s)$ is appearing in the denominator.

- Step-4

Take the gcd of all the numerators of minors the are computed in step-3 & then performed adjustment as in step-4.
↑ say this polynomial as $\psi(s)$.

The zeros are the roots of $\psi(s)$.

$$\psi(s) = s-4$$

There is one zero, which is at 4.

Pole-Zero Computation (using Smith-Mcmillan form):

$$V^{-1}AV \longrightarrow \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \; \lambda_3 \end{bmatrix}$$

Unimodumal matrix: A polynomial matrix $U(s)$ whose determinant is a non-zero scalar.

← transfer function matrix
For a given $G(s)$ one can perform row & colum operators & can obtain $\tilde{G}(s)$.

$$\tilde{G}(s) = U(s)\, G(s)\, V(s) = \begin{bmatrix} \frac{\alpha_1(s)}{\beta_1(s)} & & & & & \\ & \frac{\alpha_2(s)}{\beta_2(s)} & & & 0 & \\ & & \ddots & & & \\ & 0 & & \frac{\alpha_r(s)}{\beta_r(s)} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}$$

$r \to$ normal rank of $G(s)$

Where $U(s)$ & $V(s)$ are unimodular matrices &

(i) $\alpha_i(s)$ devides $\alpha_{i+1}(s)$ ← polynomials

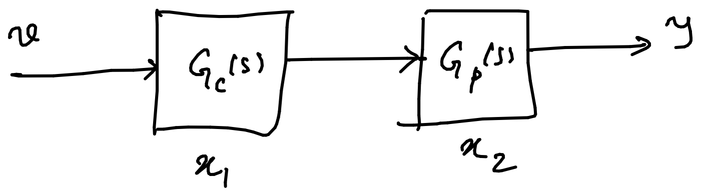
(ii) $\beta_{i+1}(s)$ devides $\beta_i(s)$

Then the roots of the polynomials $\alpha_i(s)$ are zeros of $G(s)$

The roots of $\beta_i(s)$ are the poles of $G(s)$.

→ Hidden modes / Pole-zero Cancellation :

$$G_p(s) = \frac{1}{s-1} \qquad G_c(s) = \frac{s-1}{s+1}$$



T.f from v to y

$$G(s) = \frac{1}{s+1}$$

↓

$$y = [0 \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad e\;)\;e^{x_2 t}$$

$$\begin{cases} \dot{x}_1 = -x_1 - 2v \\ \dot{x}_2 = x_2 + x_1 + v \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} v \quad ✓$$

$$y = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$t.f\,(A, b, c, d) \longrightarrow \frac{s-1}{(s-1)(s+1)} \quad ✓$$

$$x(t) = (\;)\,e^{\lambda_1 t} + (\;)\,e^{\lambda_2 t}$$

✓

λ_1 & λ_2 are the eigenvalues of A

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

The mode $\lambda_1 = 1$ is controllable or observable?

$$w_1^T A = \lambda_1 w_1^T \quad \rightarrow \quad w_1^T = [0.5 \quad 1]$$

$$A v_1 = \lambda_1 v_1 \quad \rightarrow \quad v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For controllability

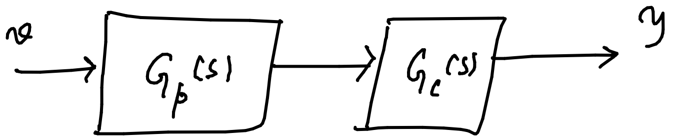
$$w_1^T b = \begin{bmatrix} 0.5 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 0$$

the mode $\lambda_1 = 1$ is <u>uncontrollable</u>.

For observability :

$$C v_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$$

$\lambda_1 = 1$ is observable mode.



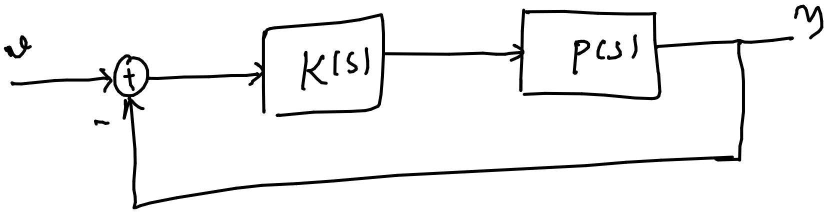
↳ $\lambda_1 = 1$ is controllable mode but <u>not observable</u>.

- <u>Hidden modes</u> of a transfer function $G(s)$ are the <u>uncontrollable</u> & <u>unobservable</u> modes.

(any uncontrollable / unobservable modes are hidden modes)

- One should be careful while cancelling the poles & zeros.

(the connection matters whether the mode is uncontrollable / unobservable.



$$K(s) = \frac{s-1}{s+1} \qquad P(s) = \frac{1}{s-1}$$

t.f. fn

$$r \text{ to } y = G(s) = \frac{PK}{1+PK}$$

$$= \frac{\frac{1}{\cancel{s-1}} \cdot \frac{\cancel{s-1}}{s+1}}{1 + \frac{1}{\cancel{s-1}} \cdot \frac{\cancel{s-1}}{s+1}}$$

$$= \frac{1}{s+2}$$