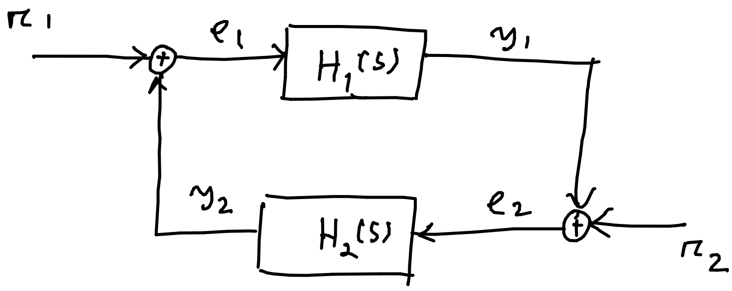
Lecture - 8 FEEDBACK INTERCONNECTION



When this feedback interconnection is "well-posed" or "well-defined"

$$H_1(s) = \frac{s-1}{s+2} \qquad H_2(s) = 1$$

$$\left[\frac{1}{1 - H_1(s)H_2(s)}\right] = \frac{s+2}{3} \leftarrow \text{improper t.f}$$

Improper t.f. are not physically realizable using analog circuits

It is not possible to write in $\begin{cases} \dot{x} = Ax + bu \\ y = cx \end{cases}$ form.

$$H_1 = \frac{s+2}{s+1} \qquad H_2 = \frac{s+1}{s+2}$$

$$\frac{1}{1 - H_1 H_2} = \frac{1}{0}$$

External / input signals : r_1 & r_2

internal / output signals : e_1, y_1, e_2 & y_2

t.f for r_1 to e_1 $= \dfrac{1}{1 - H_1 H_2}$

r_1 to y_1 $= \dfrac{H_1}{1 - H_1 H_2}$

For the feedback system input-output map

$$\begin{bmatrix} e_1 \\ e_2 \\ y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} (1 - H_1 H_2)^{-1} & (1 - H_1 H_2)^{-1} H_2 \\ (1 - H_1 H_2)^{-1} H_1 & (1 - H_1 H_2)^{-1} \\ (1 - H_1 H_2)^{-1} H_1 & (1 - H_1 H_2)^{-1} H_1 H_2 \\ (1 - H_1 H_2)^{-1} H_2 H_1 & (1 - H_1 H_2)^{-1} H_2 \end{bmatrix}}_{\substack{\text{closed loop t.f} \\ H(s)}} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

There are 4 different ~~transfer~~ function

$(1 - H_1 H_2)^{-1}$, $(1 - H_1 H_2)^{-1} H_1$, $(1 - H_1 H_2)^{-1} H_2$, $(1 - H_1 H_2)^{-1} H_1 H_2$

$(1 - H_1 H_2)^{-1} \leftarrow$ plays a crucial role.

Some Notations:

$\mathbb{R}[s] \leftarrow$ the set of polynomials having coefficients in $\mathbb{R}$

$$\mathbb{R}(s) := \left\{ g(s) \;\middle|\; g(s) = \frac{n(s)}{d(s)},\ n(s) \in \mathbb{R}[s],\ d(s) \in \mathbb{R}[s],\ d(s) \neq 0 \right\}$$

$\uparrow$ set of real rational functions

$$\boxed{\delta_r(g(s)) := \deg(d(s)) - \deg(n(s))}$$

If $\delta_\infty(g(s)) \geq 0$: $g(s)$ is proper real rational funct?

$\delta_\infty(g(s)) > 0$: $g(s)$ is strictly proper

$\delta_\infty(g(s)) = 0$: $g(s)$ is biproper

$\delta_\infty(g(s)) < 0$: $g(s)$ is improper

$$\underbrace{\frac{s+1}{s+3} \qquad \frac{1}{s+3}}_{\text{proper}} \qquad \frac{s+1}{3}$$

biproper $\qquad$ strictly proper $\qquad$ improper

$\lim_{s \to \infty} (g(s))$ when $g(s)$ are as above.

$\lim_{s \to \infty} \frac{s+1}{s+3} = 1$

$\lim_{s \to \infty} \frac{1}{s+3} = 0$

proper $\rightarrow \lim_{s \to \infty} g(s) = d$ when d is a real number.

$\times \ \lim_{s \to \infty} \frac{s+1}{3} = \infty$ ← improper t.f.

$\lim_{s \to \infty} g(s) = d \qquad d \in \mathbb{R}$ ← proper

If $d = 0$ then $g(s)$ is strictly prop

$d \neq 0$ then $g(s)$ is biproper

$$\frac{s+1}{s+3} = 1 + (\tilde{g}(s))$$

↳ strictly proper.

$(1 - H_1 H_2)^{-1}$, $(1 - H_1 H_2)^{-1} H_1$, $(1 - H_1 H_2)^{-1} H_2$, $(1 - H_1 H_2)^{-1} H_1 H_2$

$(1 - H_1 H_2)$: inverse exists & inverse is a proper transfer fnct?

For well-defined feedback interconnection we need : Above 4 tfs to be proper.

$$(1 - H_1 H_2)^{-1} \longrightarrow \frac{s+1}{s-1} \rightarrow \frac{s-1}{s+1} \quad (\;\;)$$

$$H_1(s) \rightarrow \begin{cases} \dot{x}_1 = A_1 x_1 + b_1 u_1 \\ y_1 = c_1 x_1 + d_1 u_1 \end{cases}$$

$$H_2(s) \rightarrow \begin{cases} \dot{x}_2 = A_2 x_2 + b_2 u_2 \\ y_2 = c_2 x_2 + d_2 u_2 \end{cases}$$

$$\lim_{s \to \infty} H_1(s) = d_1 \rightarrow H_1(s) = \underbrace{c_1 (sI - A_1)^{-1} b_1}_{\text{strictly proper.}} + \underline{\underline{d_1}}^{\in \mathbb{R}}$$

→ Well-posed : If all the above 4 t.fs are proper (physically realizable).

$$\Updownarrow$$

The t.f $(1 - H_1 H_2)^{-1}$ exists & it is proper.

→ Result : The feedback interconnection is well-posed iff

$$\lim_{s\to\infty} (1 - H_1(s)H_2(s)) \neq 0 \equiv (1 - d_1 d_2) \neq 0$$

$((1 - d_1 d_2)$ is invertible$)$.

For well-posedness condition is equivalent to

$(1 - H_1 H_2)^{-1}$ exists & is proper.

① $(1 - H_1 H_2)^{-1}$ exists & is proper $\Rightarrow 1 - d_1 d_2 \neq 0$

② $1 - d_1 d_2 \neq 0 \Rightarrow (1 - H_1 H_2)^{-1}$ exists & is proper

Above two statements we have to show.

① Let $(1 - H_1 H_2)^{-1}$ exists & is proper.

Existence of $(1 - H_1 H_2)^{-1} \Rightarrow$ numerator of $(1 - H_1 H_2)$ is not zero polynomial.

(*) $(1 - H_1 H_2)^{-1}$ is proper $\Rightarrow$ The degree of numerator of $(1 - H_1 H_2)^{-1}$ is less or equal to degree of denominator of $(1 - H_1 H_2)^{-1}$.

(**) Since H_1 & H_2 are proper, $(1 - H_1 H_2)$ is also proper.
⇒ the degree of numerator of $1 - H_1 H_2$ is less or equal to the degree of denominator of $1 - H_1 H_2$

$$\left(1 - H_1 H_2 = \left(1 - \frac{b}{a} \cdot \frac{y}{x}\right) = \frac{ax - by}{ax} \qquad \deg(by) \leq \deg(ax)\right)$$

From (*) & (**) statements it is clear that $(1-H_1H_2)^{-1}$ is biproper, hence it can be written as $(1-H_1H_2)^{-1} = \Delta + \underbrace{\hat{h}(s)}_{\text{strictly proper}}$,

where

$$\Delta = \frac{1}{1-d_1d_2}$$

$$\frac{2s+1}{s+3} = \underline{\underline{2}} + \left(\frac{-5}{s+3}\right)$$

$$\frac{s+1}{2s+3} = \underline{\underline{\tfrac{1}{2}}} + \left(\frac{-1/2}{2s+3}\right)$$

Hence $1-d_1d_2 \neq 0$.

- <u>second statement</u> (2)

Let $1-d_1d_2 \neq 0$. Then

$$(1-H_1H_2) = (1-d_1d_2) + \underbrace{\hat{g}(s)}_{\text{strictly proper}}$$

Let us write $\hat{g}(s) = \dfrac{\hat{b}(s)}{\hat{a}(s)}$ when $\deg(\hat{b}(s)) < \deg(\hat{a}(s))$

$$1-H_1H_2 = (1-d_1d_2) + \frac{\hat{b}(s)}{\hat{a}(s)} = \frac{(1-d_1d_2)\,\hat{a}(s) + \hat{b}(s)}{\hat{a}(s)}$$

Since $1-d_1d_2 \neq 0$, the coefficient of highest degree s term in the numerator of $1-H_1H_2$ is not zero. Hence $(1-H_1H_2)^{-1}$ exists.

Further, since $\deg(\hat{a}(s)) > \deg(\hat{b}(s))$, and $1-d_1d_2 \neq 0$, the degree of numerator of $1-H_1H_2$ is equal to the degree of denominator of $1-H_1H_2$. Hence $1-H_1H_2$ is biproper $\Rightarrow (1-H_1H_2)^{-1}$ is proper, infact it is biproper. ∎

$$H_1(s) = \begin{cases} \dot{x}_1 = A_1 x_1 + b_1 u_1 \\ y_1 = c_1 x_1 + d_1 u_1 \end{cases} \quad H_2(s) = \begin{cases} \dot{x}_2 = A_2 x_2 + b_2 u_2 \\ y_2 = c_2 x_2 + d_2 u_2 \end{cases}$$

$$e_1 = r_1 + y_2 \qquad e_2 = r_2 + y_1 \qquad \begin{matrix} u_1 = e_1 \\ u_2 = e_2 \end{matrix}$$

$$\dot{x}_1 = A_1 x_1 + b_1 u_1 \quad \text{——— (1)}$$

$$= A_1 x_1 + b_1 e_1 = A_1 x_1 + b_1 (r_1 + y_2) \quad \text{— (2)}$$

$$\dot{x}_2 = A_2 x_2 + b_2 e_2 \quad \text{——— (3)}$$

$$= A_2 x_2 + b_2 (r_2 + y_1) \quad \text{— (4)}$$

closed loop

↳ state $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ input $= \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

$$e_1 = r_1 + y_2$$

$$= r_1 + c_2 x_2 + d_2 e_2$$

$$\Rightarrow e_1 - d_2 e_2 = c_2 x_2 + r_1 \quad \text{——— (*)}$$

$$e_2 = r_2 + y_1$$

$$= r_2 + c_1 x_1 + d_1 e_1$$

$$\Rightarrow -d_1 e_1 + e_2 = r_2 + c_1 x_1 \quad \text{——— (**)}$$

From (*) & (**)

$$\begin{bmatrix} 1 & -d_2 \\ -d_1 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & c_2 \\ c_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 1 & -d_2 \\ -d_1 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 & c_2 \\ c_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \right)$$

- The feedback interconnection is <u>well-posed</u> if the internal signals e_1, e_2, y_1, y_2 can uniquely be determined for every choice of system state x & the external/input signals r_1, r_2.

$$y_1 = c_1 x_1 + d_1 e_1$$
$$= c_1 x_1 + d_1 (r_1 + y_2) \quad \text{— (i)}$$
$$\Rightarrow y_1 - d_1 y_2 = c_1 x_1 + d_1 r_1$$

$$y_2 = c_2 x_2 + d_2 e_2$$
$$= c_2 x_2 + d_2 (r_2 + y_1)$$
$$\Rightarrow -d_2 y_1 + y_2 = c_2 x_2 + d_2 r_2 \quad \text{— (ii)}$$

From (i) (ii)

$$\begin{bmatrix} 1 & -d_1 \\ -d_2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -d_1 \\ -d_2 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \right)$$

So e_i & y_i for $i = 1, 2$ can uniquely be determined for given a state x & external signal r_1 & r_2 iff

$$\begin{bmatrix} 1 & -d_1 \\ -d_2 & 1 \end{bmatrix} \text{ invertible} \Leftrightarrow 1 - d_1 d_2 \neq 0$$

Observe that in both cases (state space and t.f) the condition for well-posedness is same.

i.e.

feedback interconnection is well-posed iff $1 - d_1 d_2 \neq 0$ or $1 - d_1 d_2$ is invertible.

One can similarly extend all the above results to multi-input multi-output system.