Lecture - 9

- Let say $H_1(s)$ & $H_2(s)$ are proper.

Then

(i) $H_1(s) + H_2(s)$ is proper.

(ii) $H_1(s) \cdot H_2(s)$ is proper.

→ The feedback interconnection is well-posed iff $1 - d_1 d_2 \neq 0$.

Most of the physical plant transfer functions are strictly proper.

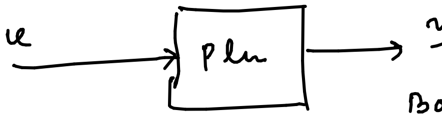
$\Downarrow$

$$\lim_{s \to \infty} G(s) = 0 \qquad G(s) = C(sI - A)^{-1}B + \cancel{D} \to 0$$

$$1 - 0 \cdot d_2 \neq 0$$

→ **Stability** :

- internally stable
- BIBO stable / Externally stable



Bounded input signal

Bounded output signal

$\Downarrow$

$\exists$ a positive number M

s.t. $|u(t)| \leq M$ ✓ $\quad \|u(t)\|_\infty \leq M$

↳ $\in L_\infty$

For an LTI system having t.f $G(s)$

BIBO stable $\Leftrightarrow$ the poles are in the open left half of $\mathbb{C}$.

For transfer funct^n matrix case, this is also true.

$\rightarrow$ Internally stable $\quad G(s) \xrightarrow{S.S.R.} \{A, b, c, d\}$

$$\dot{x} = Ax + bu \qquad \rightarrow \qquad \dot{x} = \underline{A}x + bu$$
$$y = cx + du$$

For an autonomous system:

$$\dot{x} = Ax$$

Internally stable $\equiv$ $x(t) \longrightarrow 0$ from every non-zero initial condition $x(t_0)$

✓ $x(t) = (.)e^{\lambda_1 t} + (.)e^{\lambda_2 (t)} + \cdots + (.)e^{\lambda_n (t)}$

(when A is diagonalizable)

$x(t) \longrightarrow 0$ iff $Re(\lambda i) < 0$.

- Stability
  - BIBO Stable
    T.F.
  - Internally stable
    S.S.R

- Internally stable $\Rightarrow$ BIBO stable

$$G(s) = c(sI - A)^{-1}b = \frac{c\,\text{adj}(sI-A)\,b}{|sI-A|} = \frac{b(s)}{a(s)}$$

The roots of the polynomial $|sI - A|$

are the poles of $G(s)$.

- The roots of $|sI-A|$ are the eigenvalues of A.

- If the eigenvalues of A are stable

  then the poles of $G(s)$ are also stable

  belong to open left half of $\mathbb{C}$.

  $\Downarrow$

  $G(s)$ BIBO stable.

- BIBO stable $\not\Rightarrow$ Internally stable

Eigenvalues of A — Controllable, Observable + Uncontrollable, Unobservable

$$\frac{s-1}{s-2} \rightarrow \frac{(s-1)(s-3)(s-4)}{(s-2)(s-3)(s-4)}$$

A size is 1×1

S.S.R $\rightarrow$ size of $A \rightarrow 3\times3$

- The cancelled poles with zeros are uncontrollable and/or unobservable.

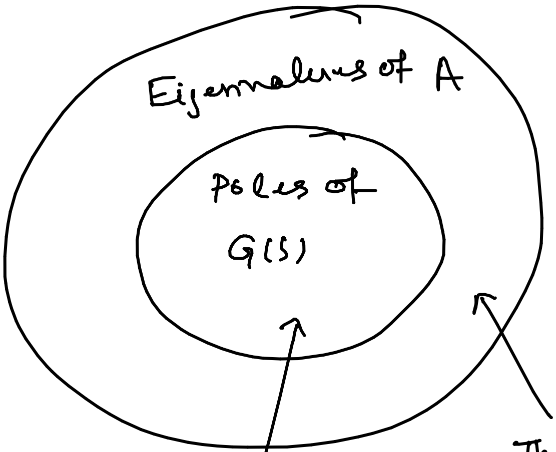
$\rightarrow$ Hence some of the the eigenvalues of A may not appear as poles of $G(s)$ (after doing cancellation)

$\hookrightarrow$ the disappeared eigenvalues of A are uncontrollable and/or unobservable

- The poles of $G(s) \subseteq$ Eigenvalues of A

So BIBO stability $\Rightarrow$ the poles of $G(s)$ are stable

$G(s) = c(sI-A)^{-1}b$

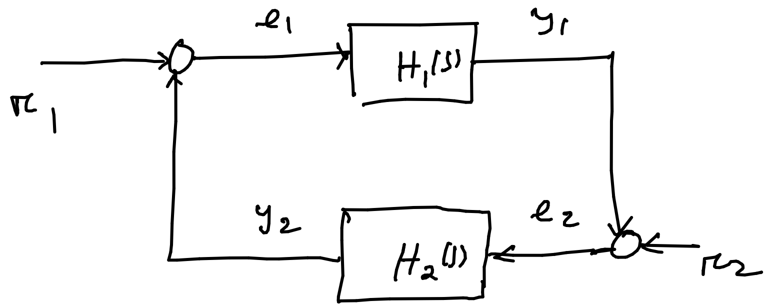


The complex number are controllable and observable

The complex numbers which are uncontrollable and/or unobservable.

BIBO stability $\not\Rightarrow$ Internal stability.

→ <u>Internal stability of feedback interconnection</u> :



$H_1(s)$
$\downarrow$ SSR

$$\left.\begin{aligned} \dot{x}_1 &= A_1 x_1 + b_1 e_1 \\ y_1 &= c_1 x_1 + d_1 e_1 \end{aligned}\right\} n \text{ dim}$$

$H_2(s)$
$\downarrow$

$$\left.\begin{aligned} \dot{x}_2 &= A_2 x_2 + b_2 e_2 \\ y_2 &= c_2 x_2 + d_2 e_2 \end{aligned}\right\} m \text{ dim}$$

The overall state $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, input $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

outputs $\begin{bmatrix} e_1 \\ e_2 \\ y_1 \\ y_2 \end{bmatrix}$

$$\begin{cases} \dot{x}_1 = A_1 x_1 + b_1 e_1 \\ \dot{x}_2 = A_2 x_2 + b_2 e_2 \end{cases}$$

$$\begin{aligned} e_1 &= r_1 + y_2 \\ &= r_1 + c_2 x_2 + d_2 e_2 \end{aligned} \qquad \begin{aligned} e_2 &= r_2 + y_1 \\ &= r_2 + c_1 x_1 + d_1 e_1 \end{aligned}$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & c_2 \\ c_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & d_2 \\ d_1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -d_2 \\ -d_1 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & c_2 \\ c_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 1 & -d_2 \\ -d_1 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 & c_2 \\ c_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \right)$$

(Assumption: the feedback interconnection is well-posed)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$= \underbrace{\left[\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} 1 & -d_2 \\ -d_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & c_2 \\ c_1 & 0 \end{bmatrix} \right]}_{A_{cl}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\left(\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} 1 & -d_2 \\ -d_1 & 1 \end{bmatrix}^{-1} \right)}_{B_{cl}} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

For internal stability → the system is autonomous

Let $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → Initial state of the closed loop system $\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$

$x_{10} \in \mathbb{R}^n$

$x_{20} \in \mathbb{R}^m$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A_{cl} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Internal Stability of FI :

Definition: The feedback interconnection is internally stable if it is well-posed and the state trajectory $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \longrightarrow 0$, as $t \to \infty$, starting from every initial condition $\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \in \mathbb{R}^{n+m}$, while setting the external inputs $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 0$.

- The feedback interconnection (FI) is internally stable iff the eigenvalues of A_{cl} are in the open left half of the complex plane.

In the FI the external signals $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ & internal signals $\begin{bmatrix} e_1 \\ e_2 \\ y_1 \\ y_2 \end{bmatrix}$.

BIBO stability : When r_1 & r_2 are bounded, the internal signals e_1, e_2, y_1 & y_2 are also bounded.

→ Assume that $\{A_1, b_1, c_1, d_1\}$ & $\{A_2, b_2, c_2, d_2\}$ are stabilizable and detectable realizations of $H_1(s)$ & $H_2(s)$, respectively. Then, the FI is internally stable iff all of the 4 tfs :

$$(1 - H_1 H_2)^{-1}, \quad (1 - H_1 H_2)^{-1} H_1, \quad (1 - H_1 H_2)^{-1} H_2 \text{ and } (1 - H_1 H_2)^{-1} H_1 H_2$$

are proper and stable.

Proof

(i) Internal stability ⇒ all of the above 4 t.fs are proper and stable

✓ (ii) All of the 4 tfs are proper and stable ⇒ Internal stable.

→ First part : Assume that FI interconnection is internally stable ⇒ (i) FI is well-posed
(ii) Eigenvalues of A_{cl} are stable.

Since FI is well-posed $\Rightarrow$ $1 - d_1 d_2 \neq 0$

$(1 - H_1(s) H_2(s))^{-1}$ exists and it is proper.

Hence all of the 4 tf.s are proper.

Since the poles of all the 4 t.f. are subset of eigenvalues of A_{cl}, they are also <u>stable</u>.

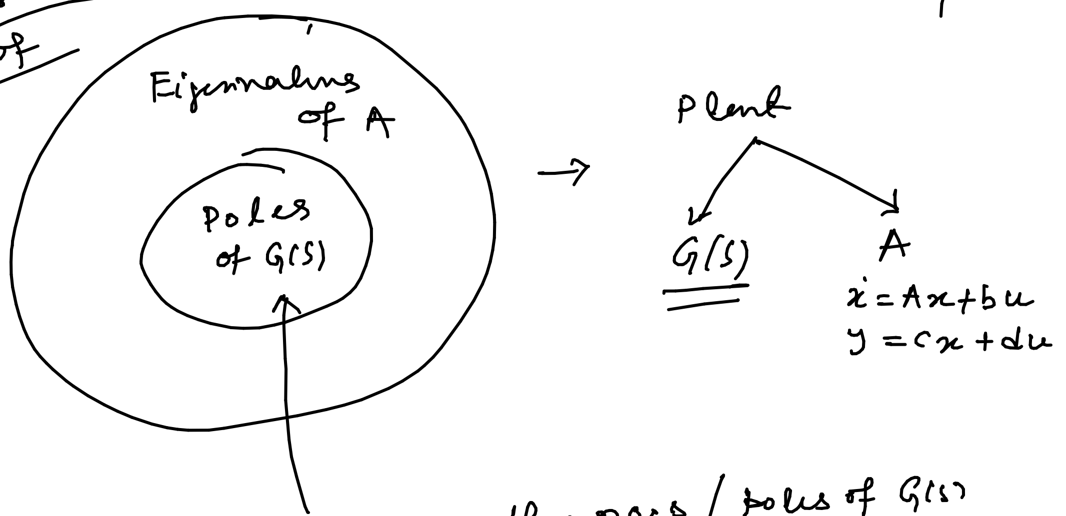
<u>2nd statement</u>

4 tfs are proper and stable.

$\Downarrow$ the system is <u>internally stable</u>.

A_{cl} is stable
&
FI is well-posed

<u>Rough sketch of proof</u>



These complex nos / poles of G(s) are stable.

We need to prove that the eigenvalues of A are stable.

- When both sets are equal $\equiv$ there is no pole zero cancellation $\equiv$ there are no uncontrollable and/or unobservable modes

Recall (A, b) pair is stabilizable

$\Downarrow$

the uncontrollable modes are stable

(C, A) pair is detectable

$\Downarrow$

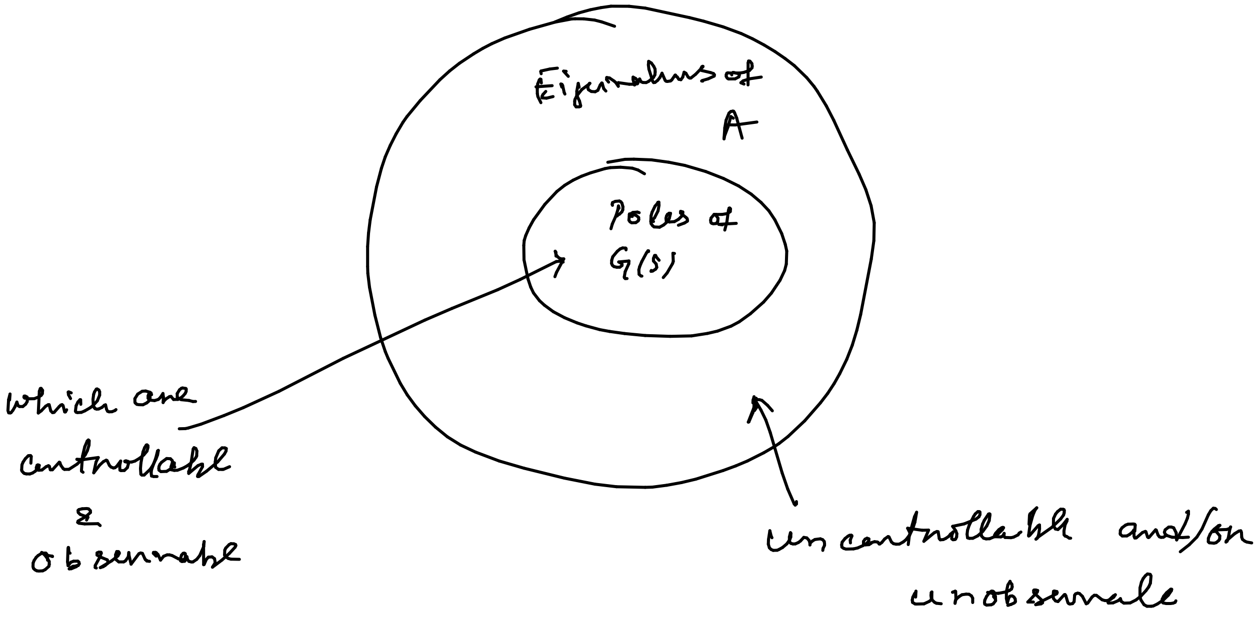
the unobservable modes are stable

- If (A, b) is stabilizable & (c, A) is detectable then all of the uncontrollable and unobservable modes of A are stable.

- then the uncontrollable & unobservable poles, which are supposed to get cancelled in G(s), are all stable.

- Further, since G(s) is stable $\Rightarrow$ the controllable and observable modes of A are also stable.

$\Rightarrow$ Hence, A is stable.



→ The 4 t.f. are stable ⇒

$$\begin{bmatrix}0 & C_2\\ C_1 & 0\end{bmatrix}\begin{bmatrix} sI - A_{cl}\end{bmatrix}\begin{bmatrix}B_1 & 0\\ 0 & B_2\end{bmatrix}$$

is also stable

Show that the pair

$$\left(A_{cl}, \begin{bmatrix}B_1 & 0\\ 0 & B_2\end{bmatrix}\right) \text{ is stabilizable}$$

$$\left(\begin{bmatrix}0 & C_2\\ C_1 & 0\end{bmatrix}, A_{cl}\right) \text{ is detectable.}$$

⇒ A_{cl} is stable.