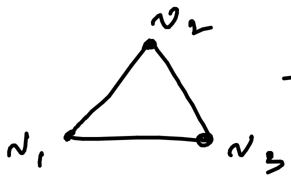
ELL 805

Lecture - 10

Result: Let G be an undirected graph with 'n' vertices and 'm' edges. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of adjacency matrix $A(G)$. Then

$$\boxed{\lambda_1 \leq \sqrt{\frac{2m(n-1)}{n}}}$$

Proof

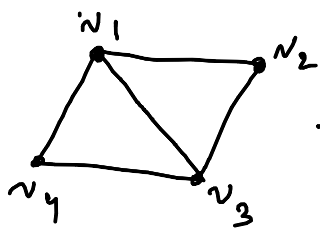


$$\rightarrow A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$d(v_i) = 2$

$\sum d(v_i) = 2 \times 3$

No of edges



$$\rightarrow A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\downarrow$$

$$A^2 = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \rightarrow$$

$d(v_1) = 3$

$d(v_2) = 2$

$d(v_3) = 3$

$d(v_4) = 2$

$\sum d(v_i) = 10$

$= 2 \times 5$

No of edges

- The diagonal entries of A^2 are

$$\boxed{[A^2]_{ii} = d(v_i)}$$

for $i = 1, 2 \cdots n$

- $$\boxed{\sum_{i=1}^{n} d(v_i) = 2m}$$

- Trace of $A \longrightarrow$

$$tr(A) = \sum_{i=1}^{n} [A]_{ii}$$
$$= \lambda_1 + \lambda_2 + \cdots + \lambda_n$$
$$= 0$$

For a given graph G with Adjacency matrix $A(G)$ we have $\boxed{\sum_{i=1}^{n} \lambda_i = 0}$ . . . *

- $$tr(A^2) = \sum_{i=1}^{n} [A^2]_{ii}$$

eig(A^2) and λ_i^2

$$= \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2$$
$$= d(v_1) + d(v_2) + \cdots + d(v_n)$$
$$= 2m$$

$$\boxed{\Rightarrow \quad \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2 = 2m} \; \ldots \; (**)$$

From (*) $\quad \lambda_1 + \lambda_2 + \cdots + \lambda_n = 0$

$$\Rightarrow \lambda_1 = -(\lambda_2 + \lambda_3 + \cdots + \lambda_n)$$

$$\Rightarrow |\lambda_1| = |\lambda_2 + \lambda_3 + \cdots + \lambda_n|$$

triangle inequality

$$\leq |\lambda_2| + |\lambda_3| + \cdots + |\lambda_n| \; \ldots \; (\Delta)$$

From (**)

$$\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2 = 2m$$

$$\Rightarrow 2m - \lambda_1^2 = \lambda_2^2 + \lambda_3^2 + \cdots + \lambda_n^2 \quad \cdots\cdots (\Delta\Delta)$$

x

Cauchy-Schwartz Inequality

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\left(\sum_{i=1}^{n} x_i \cdot y_i \right)^2 \leq \left(\sum_{i=1}^{n} x_i^2 \right)\left(\sum_{i=1}^{n} y_i^2 \right)$$

or

$$|x^T y|^2 \leq \|x\|_2 \|y\|_2$$

x

Let us define $x = \begin{bmatrix} |\lambda_2| \\ |\lambda_3| \\ \vdots \\ |\lambda_n| \end{bmatrix}$ $y = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n-1}$

According to Cauchy-Schwartz inequality

$$\left(|\lambda_2| + |\lambda_3| + \cdots + |\lambda_n| \right)^2 \leq \left(\sum_{i=2}^{n} |\lambda_i|^2 \right)(n-1)$$

$$\Rightarrow \sum_{i=2}^{n} |\lambda_i|^2 \geq \frac{1}{n-1} \left(\sum_{i=2}^{n} |\lambda_i| \right)^2 \quad \cdots\cdots (\square)$$

For (ΔΔ), we have

$$2m - \lambda_1^2 = \lambda_2^2 + \lambda_3^2 + \cdots + \lambda_n^2$$

$$= |\lambda_2|^2 + |\lambda_3|^2 + \cdots + |\lambda_n|^2$$

$$\geq \frac{1}{n-1}\left(|\lambda_2| + |\lambda_3| + \cdots + |\lambda_n|\right)^2 \quad (\text{From } \mathrm{II})$$

$$\geq \frac{1}{n-1}|\lambda_1|^2 \quad (\text{from } \Delta)$$

$$\Rightarrow \quad 2m \geq \lambda_1^2\left(1 + \frac{1}{n-1}\right) = \lambda_1^2\left(\frac{n}{n-1}\right)$$

$$\Rightarrow \quad \lambda_1^2 \leq \frac{2m(n-1)}{n}$$

$$\Rightarrow \quad \lambda_1 \leq \sqrt{\frac{2m(n-1)}{n}}$$

- The above result gives a bound on the largest eigenvalue of adjacency matrix of a graph in terms of the number of vertices & edges of the graph.

* Derive a similar expression for Laplacian matrix.

↑ Exercise.