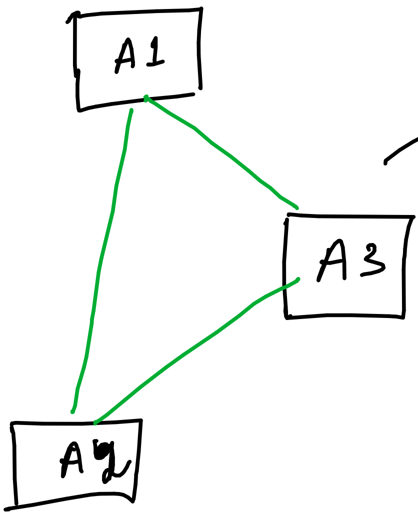
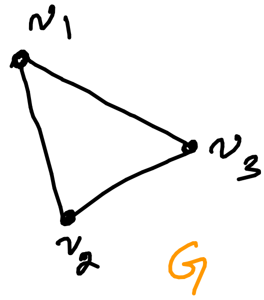
ELL805

Lecture - 11

→ **Consensus Problem**





Let the dynamics of each agent is

$\dot{x}_i = u_i$ $\qquad$ $x_i(0)$ is some initial condition

x: position of a robot (A_i)

Objective : All the states of the agents converge to a common point (starting with different initial conditions.

$\dot{x}_i = u_i$

Consider the above n/w topology

Neighbor of agent - 1 : $\{2, 3\}$

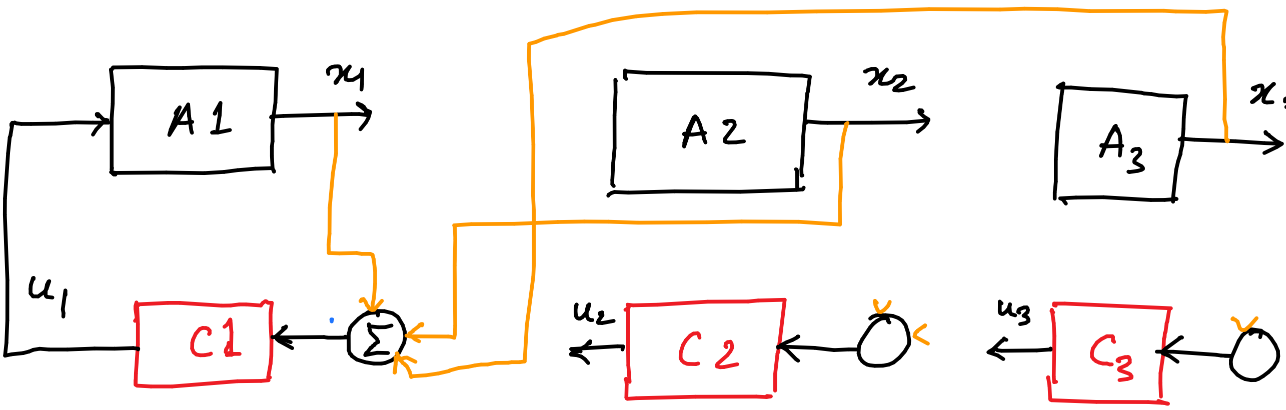
↓

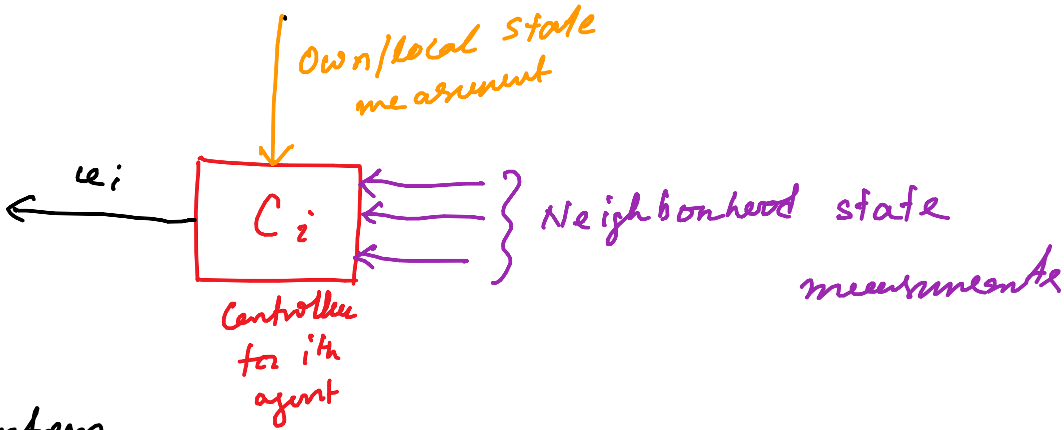
Let us consider the following control protocol

$$u_1 := (x_2 - x_1) + (x_3 - x_1) = -2x_1 + x_2 + x_3$$

$$u_2 := (x_3 - x_2) + (x_1 - x_2) = -2x_2 + x_1 + x_3$$

$$u_3 : (x_1 - x_3) + (x_2 - x_3) = -2x_3 + x_1 + x_2$$

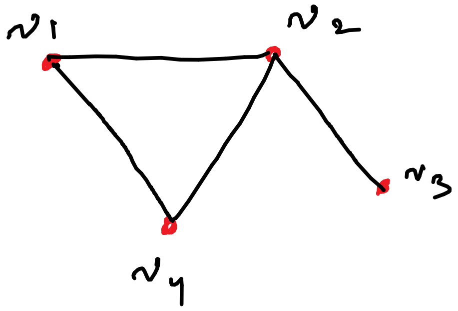




closed loop system

$$\left.\begin{aligned} \dot{x}_1 &= -2x_1 + x_2 + x_3 \\ \dot{x}_2 &= -2x_2 + x_1 + x_3 \\ \dot{x}_3 &= -2x_3 + x_1 + x_2 \end{aligned}\right\} \rightarrow \overset{\dot{x}}{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}} = \underbrace{\overset{A}{\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}}}_{-L(G)} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

Laplacian matrix of graph G (representing the communication topology)



$$\dot{x}_i = u_i$$

control protocol

$$\left\{\begin{aligned} u_1 &= (x_2 - x_1) + (x_4 - x_1) \\ u_2 &= (x_1 - x_2) + (x_4 - x_2) + (x_3 - x_2) \\ u_3 &= x_2 - x_3 \\ u_4 &= (x_1 - x_4) + (x_2 - x_4) \end{aligned}\right.$$

Closed loop system

$$\dot{x}_1 = -2x_1 + x_2 + x_4$$

$$\dot{x}_2 = -3x_2 + x_1 + x_2 + x_3$$

$$\dot{x}_3 = -x_3 + x_2$$

$$\dot{x}_4 = -2x_4 + x_1 + x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}}_{-L(G) \text{ or } -L} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The closed loop system

$$\underbrace{\dot{x} = -Lx}_{\text{sol}^n \text{ of}} \quad \text{with} \quad x(0) = x_0$$

Laplacian matrix $\mathcal{L} \xrightarrow[\text{orthogonal matrix}]{Q} \underbrace{\begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \lambda_{n-1} & \\ & & & & 0 \end{bmatrix}}_{\mathcal{L}_\Lambda}$

$$Q^T Q = I$$

$$\mathcal{L}_\Lambda = Q^T \mathcal{L} Q$$

Let now define a new state variable

$$z = Q^T x \Rightarrow x = Qz$$

$$\dot{x} = -\mathcal{L} x$$

$$\Rightarrow Q\dot{z} = -\mathcal{L} Q z$$

$$\Rightarrow \dot{z} = -Q^T \mathcal{L} Q z$$

$$= -\mathcal{L}_\Lambda z$$

$$\dot{z}_i = -\lambda_i z_i$$

$$\Rightarrow z_i = e^{-\lambda_i t} z_i(0)$$

$$\underbrace{\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}}_{Q^T x} = \underbrace{\begin{bmatrix} e^{-\lambda_1 t} & & & \\ & e^{-\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{-\lambda_n t} \end{bmatrix}}_{e^{\Lambda t}} \underbrace{\begin{bmatrix} z_{10} \\ z_{20} \\ \vdots \\ z_{n0} \end{bmatrix}}_{Q^T x_0}$$

$$\begin{tikzcd} x \arrow[r, bend left, "Q^T"] & z \arrow[l, bend left, "Q"] \end{tikzcd}$$

$$z = Q^T x$$

$$\Rightarrow z(0) = Q^T x(0)$$

$$\boxed{\Rightarrow x = Q e^{\Lambda t} Q^T x_0}$$

$$Q = \begin{bmatrix} | & | & & | & | \\ q_1 & q_2 & \cdots & q_{n-1} & q_n \\ | & | & & | & | \end{bmatrix} \qquad \begin{cases} q_i^T q_i = 1 \\ q_i^T q_j = 0 \quad \text{for } i \neq j \end{cases}$$

Let the eigenvalues of L are arranged

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n$$

q_n is the eigenvector of L corresponding to '0' eigenvalue.

$$q_n = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$x = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} e^{-\lambda_1 t} & & & \\ & e^{-\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{-\lambda_n t} \end{bmatrix} \underbrace{\begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix} \begin{bmatrix} x_0 \end{bmatrix}}$$

$$= \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} e^{-\lambda_1 t} & & & \\ & e^{-\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{-\lambda_n t} \end{bmatrix} \begin{bmatrix} q_1^T x_0 \\ q_2^T x_0 \\ \vdots \\ \underbrace{q_n^T x_0}_{\text{scalars}} \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} e^{-\lambda_1 t} q_1^T x_0 \\ e^{-\lambda_2 t} q_2^T x_0 \\ \vdots \\ \underbrace{e^{-\lambda_n t} q_n^T x_0}_{\text{scalars}} \end{bmatrix}$$

$$x = \underbrace{q_1 e^{-\lambda_1 t} q_1^T x_0}_{\xrightarrow{t\to\infty} 0} + \underbrace{q_2 e^{-\lambda_2 t} q_2^T x_0}_{\xrightarrow{t\to\infty} 0} + \cdots + \underbrace{q_{n-1} e^{-\lambda_{n-1} t} q_{n-1}^T x_0}_{\xrightarrow{t\to\infty} 0} + \underbrace{q_n e^{-\lambda_n t} q_n^T x_0}_{\to q_n q_n^T x_0}$$

$x(t) \to \infty$

$\lambda_i > 0$ for $i = 1, 2, \dots, n-1$

$\Downarrow$

$x(t)$ will converge to $q_n q_n^T x_0$ assymptotically

$$q_n q_n^T x_0 = \underbrace{\frac{\mathbb{1}^T x_0}{n}} \cdot \mathbb{1} \qquad q_n = \frac{1}{\sqrt{n}} \cdot \mathbb{1} .$$

$$x(t) \longrightarrow \xrightarrow{t\to\infty} \qquad x_1 = x_2 = x_3 = \cdots = x_n = \eta$$

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \underbrace{\left(\frac{x_{10} + x_{20} + x_{30}}{n}\right)}_{\eta} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$