ELL 805

Lecture- 12

Multiple agents with dynamics $\dot{x}_i = u_i$

For the use of control protocol

Also said Agreement Protocol → $\boxed{u_i = \sum_{j \in N(i)} x_j - x_i}$ .... (*)

$N(i)$: The set of neighborhood agents of ith agent

- If we use (*), then consensus (state) can be achieved.

- the closed loop dynamics

$$\dot{x} = -Lx \quad , \quad x(0) = x_0$$

$$x(t) = e^{-\lambda_1 t}(q_1^T x_0) q_1 + e^{-\lambda_2 t}(q_2^T x_0) q_2 + \cdots + e^{-\lambda_{n-1} t}(q_{n-1}^T x_0) q_{n-1} + e^{-\lambda_n t}(q_n^T x_0) q_n$$

Since $\begin{cases} \lambda_i > 0 & \text{for } i = 1, 2 \cdots n-1 \\ \lambda_n = 0 \end{cases}$

$$x(t) \xrightarrow[t \to \infty]{} (q_n^T x_0) q_n = \underbrace{\frac{\mathbb{1}^T x_0}{n}}_{\eta} \cdot \mathbb{1}$$

* If we choose the control protocol as

$$u_i = \gamma \left[\sum_{j \in N(i)} x_j - x_i \right]$$

γ is a +ve scalar.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = -\gamma L \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\dot{x} = -\underbrace{\gamma L}\, x$$

All the eigenvalues of $-L$ will be scaled by γ.

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$$

$\underbrace{e^{-\lambda_1 t}(q_1^T x_0)}_{\text{scalar}} q_1 \leftarrow$ Converge to '0' very fast in comparison to $e^{-\lambda_{n-1}}(q_{n-1}^T x_0) q_{n-1}$

- The speed of convergence of $x(t) \to \eta$ depends on the eigenvalue λ_{n-1}.
- The quantity "γ" can be used to change the speed of convergence.

→ Agreement Subspace:

$$\mathcal{A} := \{ x \in \mathbb{R}^n \mid x_i = x_j \text{ for all } i, j \}$$

↑

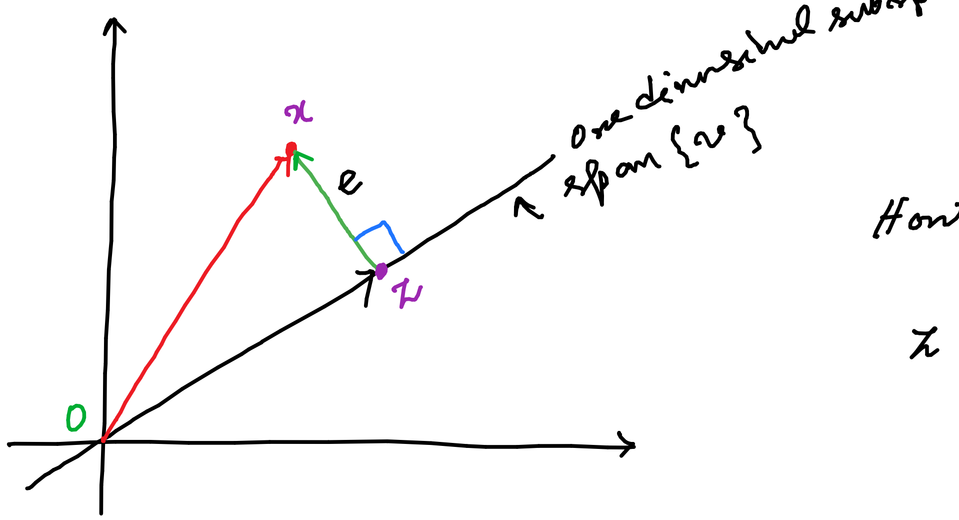
span $\{\mathbb{1}\}$

→ Using the above state control protocols $x(t)$ will converge to the agreement subspace

$$x(t) \longrightarrow \eta = \left(\frac{\mathbb{1}^T x_0}{n}\right)\mathbb{1}$$

→ <u>Orthogonal Projection</u>

Consider an one dimensional subspace span $\{v\}$



How to express z point

$$z = \alpha v \qquad \alpha : \text{is a scalar}$$

$$z + e = x \quad \Rightarrow \quad e = x - z$$

• Since the vectors z & e are orthogonal

$$z^T e = 0$$

$$\Rightarrow z^T(x - z) = 0$$

$$\Rightarrow z^T x - z^T z = 0$$

$$\Rightarrow z^T x = z^T z$$

$$\Rightarrow z^T x = \alpha^2 \underbrace{v^T v}_{\text{scalar}}$$

$$\Rightarrow \alpha^2 = \frac{1}{v^T v}(\alpha v^T x)$$

$$\boxed{\Rightarrow \alpha = \frac{v^T x}{v^T v}}$$

Since $z = \alpha v$

We have $z = \dfrac{v^T x}{v^T v} \cdot v$

- Define a matrix $\boxed{P := \dfrac{v v^T}{v^T v}}$ ← Orthogonal Projector

$$x \xrightarrow{P} z$$

$$Px = \left(\frac{v v^T}{v^T v}\right) x$$

$$= \underbrace{\frac{v^T x}{v^T v}}_{\alpha} \cdot v = z$$

- The orthogonal projector P takes the point x & projects onto the space : span$\{v\}$.

$$\Downarrow$$

Since $Px = z$

Orthogonal projection of point x on the subspace span$\{v\}$.

- P is symmetric
- $P^2 = P$

$$P^2 = \left(\frac{v v^T}{v^T v}\right)\left(\frac{v v^T}{v^T v}\right) = \frac{v v^T}{v^T v} = P$$

- $P^2 x = P(Px) = Pz = z$ $\qquad z = \alpha v$

- The agreement subspace that we have defined $\mathcal{A}$ is an one dimensional subspace : span $\{\mathbb{1}\}$

$v = \mathbb{1}$

Let use define an onthogonal projecton

$$\boxed{P = \frac{\mathbb{1}\mathbb{1}^T}{\mathbb{1}^T\mathbb{1}}}$$

x_0 is initial condition

$$P x_0 = \left(\frac{\mathbb{1}\mathbb{1}^T}{\mathbb{1}^T\mathbb{1}}\right) x_0$$

$$= \left(\frac{\mathbb{1}^T x_0}{\mathbb{1}^T\mathbb{1}}\right) \cdot \mathbb{1} \equiv \eta$$

$\therefore$ η is an elmt of agreement subspace $\mathcal{A}$.

- The state trajectory generated by the closed loop system $\dot{x} = -Lx$, converges to the onthogonal projection of its initial state x_0, onto the agreement subspace $\mathcal{A}$.

$$x(t) \longrightarrow \eta = \frac{\mathbb{1}^T x_0}{\mathbb{1}^T\mathbb{1}} \cdot \mathbb{1} = \left(\frac{x_{10} + x_{20} + \cdots + x_{n0}}{n}\right) \cdot \mathbb{1}$$

$\uparrow$

<u>Average Consensus</u>