ELL805

Lecture - 13

- **Non-negative Matrix (M):**

  A matrix $M \in \mathbb{R}^{n \times n}$ is said to be "non-negative" if its entries: $m_{ij} \geq 0$ for all $i,j = 1,2 \cdots n$.

- **Positive Matrix (M):**

  A matrix $M \in \mathbb{R}^{n \times n}$ is "positive" if $m_{ij} > 0$ for all $i,j = 1,2 \cdots n$.

- **Stochastic Matrix:** A non-negative matrix P is called "stochastic matrix" if its row sums are 1.

- **Doubly Stochastic matrix:** A non-negative matrix P is doubly stochastic if both row & column sums to 1.

- For stochastic matrix $P \rightarrow$ $\boxed{P \cdot \mathbb{1} = \mathbb{1}}$ $\qquad \mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

  For doubly stochastic matrix $P \rightarrow$ $\boxed{\begin{array}{c} \mathbb{1}^T P = \mathbb{1}^T \\ P\mathbb{1} = \mathbb{1} \end{array}}$

$A v_i = \lambda_i v_i \rightarrow$ λ_i & v_i are eigenvalues & eigenvectors, respectively

$\updownarrow$

$$P\mathbb{1} = \mathbb{1}$$

→ The vector $\mathbb{1}$ is an <u>eigenvector</u> corresponding to the <u>eigenvalue 1</u>.

- For doubly stochastic matrix

$$\left\{ \begin{array}{l} \mathbb{1}^T P = \mathbb{1}^T \\ P\mathbb{1} = \mathbb{1} \end{array} \right.$$

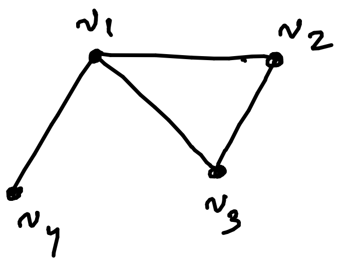
↓

Example : Permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \leftarrow \text{Doubly stochastic matrix.}$$

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 1 & 4 \\ 8 & 2 & 5 \\ 9 & 3 & 6 \end{bmatrix}$$

→ <u>Normalized Adjacency matrix</u>



$d(v_1) = 3$

$d(v_2) = 2$

$d(v_3) = 2$

$d(v_4) = 1$

Define a matrix $A_N(G)$ whose elements are

$$[A_N(G)]_{ij} = \begin{cases} \frac{1}{d(v_i)}, & \text{if } v_i \text{ \& } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

No self-loops
⇓
Diagonal entries are '0'

$$A_N(G) := \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$ ← Stochastic matrix.

→ Spectral Radius of a matrix:

$$\boxed{\rho(A) := \max_{\lambda_i} |\lambda_i|}$$

$\lambda_i \in \mathbb{C}$

$|\cdot|$ ← Modulus

- Result: The spectral radius of a stochastic matrix $\rho(P)$ is one: $\rho(P) = 1$.

Proof

We know for all stochastic matrix

$$P \cdot \mathbb{1} = \mathbb{1} \Rightarrow \|P\|_\infty = 1$$

. . . .

- For every matrix A, $\rho(A) \leq \|A\|$ for all matrix norms.

- For matrix A, we have:

$$A v_i = \lambda_i v_i$$

eigenvector eigenvalue

- Construct a matrix $X = \begin{bmatrix} 0 & 0 & \cdots & 0 & v_i & 0 & \cdots & 0 \end{bmatrix}_{n \times n}$

$X \neq 0$, since $v_i \neq 0$

ith column

$$AX = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_n^T & - \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & v_i & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & \vdots & a_1^T v_i & \vdots & 0 \\ & & a_2^T v_i & & \\ & & \vdots & & \\ & & a_n^T v_i & & \end{bmatrix}$$

$A v_i = \lambda_i v_i$

$$\Rightarrow \quad AX = \lambda_i X$$

$$\Rightarrow \quad \|\lambda_i X\| = \|AX\|$$

$$\Rightarrow \quad |\lambda_i| \, \|X\| = \|AX\| \leq \|A\| \|X\|$$ ← Matrix norm properties.

$$\Rightarrow \quad |\lambda_i| \leq \|A\|$$

$\Downarrow$

Since λ_i is any arbitrary eigenvalue of A, the above relation holds for all $i = 1, 2, \cdots n$.

$$\Rightarrow \quad \rho(A) \leq \|A\| .$$

For a stochastic matrix P

$$\|P\|_\infty = 1 \qquad \|\cdot\|_\infty : \text{Infinity norm of a matrix}$$

$$\Rightarrow \rho(P) \leq 1$$

Since 1 is an eigenvalue of $P \Rightarrow \rho(P) = 1$ ∎

→ Perron's Results

Let P be a positive matrix.

Then the following statements hold:

- $r := \rho(A) > 0$
- 'r' is an eigenvalue of P (called "Perron root".)
- Algebraic Multiplicity of r is one.
- There exists an eigenvector $x > 0$ ($x_i > 0$) s.t. $Px = rx$.
- There exists a unique vector $v > 0$, s.t.

$$Pv = rv \quad \text{and} \quad \sum_{i=1}^{n} v_i = 1$$

called "Perron vector".

→ A matrix M is said to be <u>reducible</u> matrix if there exists a permutation matrix P s.t.

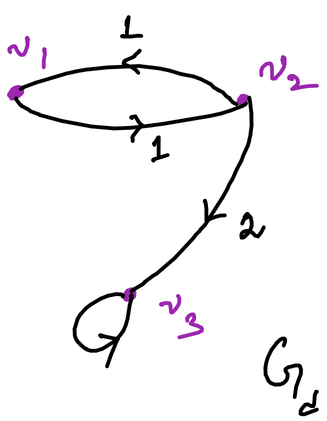
$$P^T M P = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \quad \text{(block upper } \Delta^r \text{ matrix)}$$

If the above condition does not hold then M is called "<u>irreducible</u>" matrix.

- For a given matrix $M \in \mathbb{R}^{n \times n}$, we associate a digraph G_d as follows:
  - The number of colums = no. of rows = the number of vertices of digraph G
  - there is an directed edge from vertex v_j to v_i if $m_{ij} \neq 0$.

    ↑ an elmt of M.

$$\begin{array}{c} \begin{array}{ccc} & m_{12} & \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix} \\ M \end{array} \longrightarrow$$



→ **Result**

A matrix M is irreducible if and only if the associated digraph G_d is "strongly connected".

→ **Perron - Frobenius Results**

Let $M \in \mathbb{R}^{n \times n}$ be a non-negative matrix, which is irreducible. Then following statements hold:

- $r := \rho(M) > 0$
- Algebraic multiplicity of $r = 1$
- There exists an eigenvector $x > 0$

  s.t. $Mx = rx$

- There exists a unique vector $v > 0$ s.t.

$$Mv = rv \quad \text{with} \quad \sum_{i=1}^{n} v_i = 1$$

$\uparrow$

Perron vector

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$