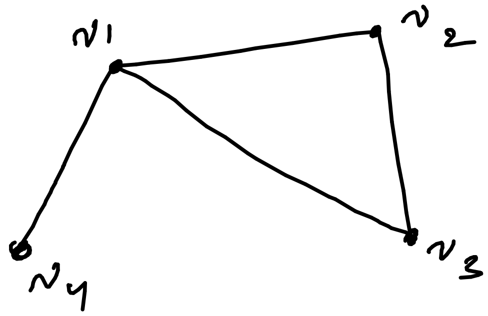
ELL805

Lecture - 14

→ Consensus in Discrete-time Agent Dynamics.

Agent Dynamics : $x_i(k+1) = x_i(k) + u_i(k)$ $\qquad x_i(k) \in \mathbb{R}$



$d(v_1) = 3$

$d(v_2) = 2$

$d(v_3) = 2$

$d(v_4) = 1$

Normalized Adjacency matrix

$$A_N = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

→ Not symmetric

Define Laplacian matrix as follow

$$\boxed{L_N := I - A_N}$$

$$L_N = \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

→ Not symmetric

→ row sum is 0.

For agent dynamics $x_i(k+1) = x_i(k) + u_i(k)$

Define the control protocol as follow

$$\boxed{u_i(k) = \frac{1}{d(v_i)}\left[\sum_{j \in \mathcal{N}(i)} \left(x_j(k) - x_i(k)\right)\right]}$$

$\mathcal{N}(i)$: Neighborhood set of ith agent.

For the considered example:

$$u_1(k) = \frac{1}{3}\left[\left(x_2(k) - x_1(k)\right) + \left(x_3(k) - x_1(k)\right) + \left(x_4(k) - x_1(k)\right)\right]$$

$$u_2(k) = \frac{1}{2}\left[\left(x_1(k) - x_2(k)\right) + \left(x_3(k) - x_2(k)\right)\right]$$

$$u_3(k) = \frac{1}{2}\left[\left(x_2(k) - x_3(k)\right) + \left(x_1(k) - x_3(k)\right)\right]$$

$$u_4(k) = x_1(k) - x_4(k)$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}}_{-L_N} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

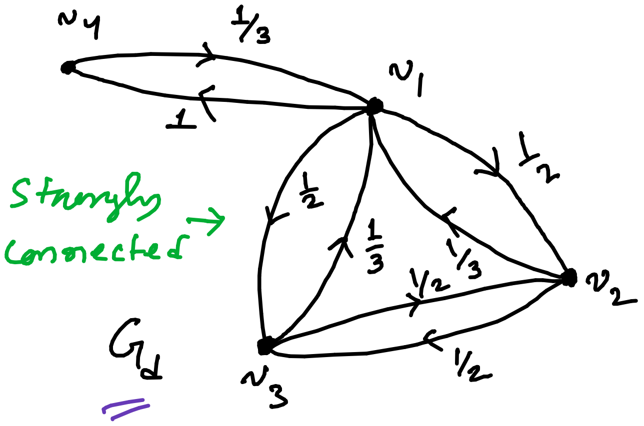
$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \underbrace{\begin{bmatrix} -L_N \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}}_{u(k)}$$

The closed loop system:

$$\boxed{\begin{aligned} x(k+1) &= (I - L_N)\,x(k) \\ &= A_N\, x(k) \end{aligned}}$$

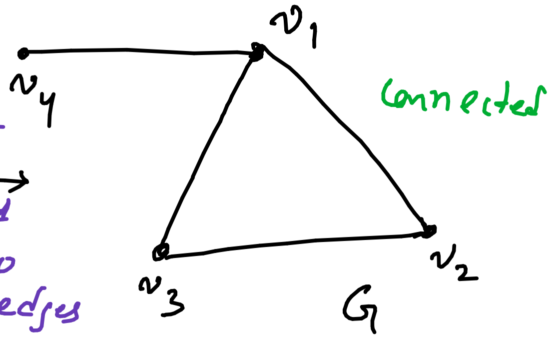
The properties of A_N:

- It is Non-negative and stochastic matrix
- Since A_N is stochastic matrix, the spectral radius $\rho(A_N) = 1$
- Let us construct a digraph corresponding to the normalized adjacency matrix A_N:

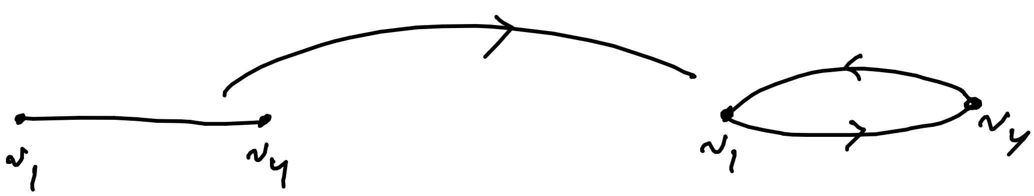


Digraph associated with A_N

Each edge in G is replaced with two directed edges in G_d



Undirected communication topology.



Part of G_d

- The digraph associated with A_H (G_d) is "strongly connected" iff the network topology G is "connected".

$$\Downarrow$$

A_H has "strongly connected" property.

$$\Downarrow$$

A_H is irreducible.
& also A_H is non-negative

$$\Downarrow \text{ Perron-Frobenius Results}$$

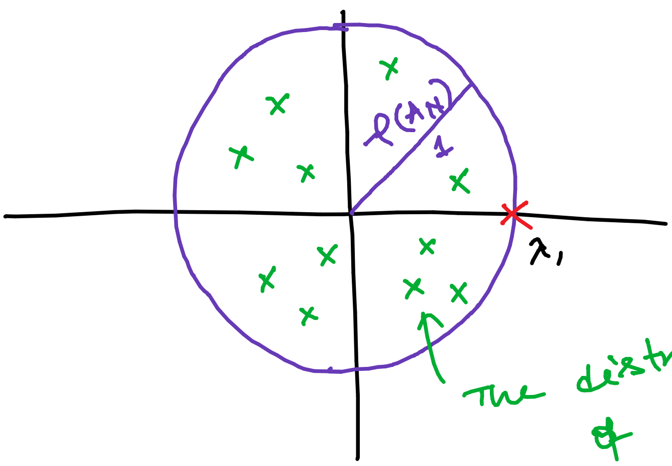
- $\rho(A_H) = 1$ is an eigenvalue of A_H.
- The Algebraic multiplicity of 1 is one.

- Since A_H is stochastic

$$A_H \cdot \mathbb{1} = \mathbb{1}$$

$$\Downarrow$$

$\mathbb{1}$ is an eigenvector of A_H corresponding to the eigenvalue 1.



The distribution of eigenvalues of A_H which are less than 1.

$$x(k+1) = A_N x(k) \quad \cdots\cdots \quad (*)$$

The initial condition $x(0)$

$$x(1) = A_N x(0)$$

$$x(2) = A_N x(1) = A_N^2 x(0)$$

$$x(3) = A_N x(2) = A_N^3 x(0)$$

$$\vdots$$

$$\boxed{x(k) = A_N^k x(0)} \leftarrow \text{solution of } (*)$$

- Let us assume that A_N has 'n' distinct eigenvalues.

$$V^{-1} A V = \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{\Lambda} \Rightarrow A = V \Lambda V^{-1}$$

Constructed by using the eigenvectors of A

→ Show that $A^k = V \Lambda^k V^{-1}$

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}$$

$$A v_i = \lambda_i v_i$$

$$A^2 v_i = \lambda_i^2 v_i$$

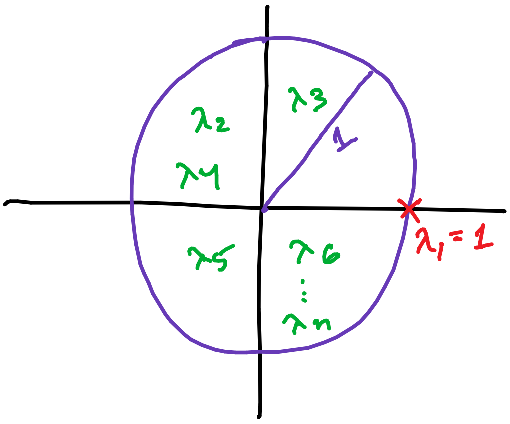
From the above we can write

$$\boxed{A_N^k = \lambda_1^k v_1 q_1^T + \lambda_2^k v_2 q_2^T + \cdots + \lambda_n^k v_n q_n^T} \quad (A_N^0 = I)$$

where v_i & q_i are the right & left eigenvectors, respectively, of A_N corresponding to eigenvalue λ_i.

The soln

Let $\underset{\substack{\| \\ 1}}{\lambda_1} > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$



The complex number λ_i can be represented as

$$\lambda_i = r e^{i\theta}$$

where $r = |\lambda_i|$

$$\Rightarrow \quad \lambda_i^k = r^k e^{ik\theta}$$

The solution

$$x(k) = A_N^k x(0)$$

$$= \Big[\underset{\substack{\| \\ 1}}{\lambda_1^k} v_1 q_1^T + \underbrace{\lambda_2^k v_2 q_2^T + \cdots + \lambda_n^k v_n q_n^T}_{\lambda_i\text{'s satisfy } |\lambda_i| < 1}\Big] x(0)$$

$$\Rightarrow \lim_{k\to\infty} x(k) = \lim_{k\to\infty} \Big[\underbrace{\lambda_1^k v_1 q_1^T}_{\downarrow\ v_1 q_1^T x(0)} + \underbrace{\lambda_2^k v_2 q_2^T + \cdots + \lambda_n^k v_n q_n^T}_{\to 0}\Big] x(0)$$

Recall that v_1 is an eigenvector of A_N corresponding to the eigenvalue 1. $\Rightarrow v_1 = \mathbb{1}$

Hence $x(k) \xrightarrow{k\to\infty} v_1 q_1^T x_0 = \underbrace{(q_1^T x_0)\cdot \mathbb{1}}_{\in \text{ span}\{\mathbb{1}\}}$

Hence $x(k)$ converges to the Agreement subspace.

$\Rightarrow$ Consensus Achieved.