ELL805

Lecture - 15

Agent Dynamics : $x_i(k+1) = x_i(k) + u_i(k)$

$$u_i(k) = \frac{1}{d(v_i)} \left[\sum_{j \in \mathcal{N}(i)} (x_j(k) - x_i(k)) \right]$$

The closed loop system :

$$x(k+1) = A_N x(k)$$

A_N : Normalized Adjacency matrix.

soln Assumption: A_N has distinct eigenvalues.

$$x(k) = \left[\lambda_1^k v_1 q_1^T + \lambda_2^k v_2 q_2^T + \cdots + \lambda_n^k v_n q_n^T \right] x(0)$$

$$\Big\downarrow k \to \infty$$

$$x(k) = v_1 q_1^T x_0 = (q_1^T x_0) \cdot \mathbb{1} \in \text{Agreement subspace.}$$

→ A_N may not be symmetric.

- There is also a possibility that A_N is not diagonalizable ! (show one such communicatn topology)

→ When we can not diagonalize A_N, we can at least transfer A_N to a block diagonal form.

- There exists a non-singular matrix P

s.t.

$$P^{-1} A_N P = J = \begin{bmatrix} J(\lambda_1) & & & 0 \\ & J(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J(\lambda_n) \end{bmatrix}$$

$\lambda_1, \lambda_2 \cdots \lambda_n$ are the distinct eigenvalues of A_N.

where

$$J(\lambda_i) = \begin{bmatrix} J_1(\lambda_i) & & & \\ & J_2(\lambda_i) & & \\ & & \ddots & \\ & & & J_m(\lambda_i) \end{bmatrix}$$

where

$$\underset{1,2\cdots m}{J_*(\lambda_i)} = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} \lambda_i & & & \\ & \lambda_i & & \\ & & \ddots & \\ & & & \lambda_i \end{bmatrix}}_{\lambda_i I} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ & & & & 1 \\ 0 & 0 & & & 0 \end{bmatrix}}_{\text{Nilpotent Matrix}}$$

Nilpotent matrices → $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

For a Nilpotent matrix : There exists a positive integer r s.t.

$$N^r = 0 \quad \& \quad N^{r-1} \neq 0$$

r : is called Nilpotent index of N.

$$J = \begin{bmatrix} 4 & 1 & 0 & & & & & & \\ 0 & 4 & 1 & & & & & & \\ 0 & 0 & 4 & & & & & & \\ & & & 4 & 1 & & & & \\ & & & 0 & 4 & & & & \\ & & & & & 3 & 1 & & \\ & & & & & 0 & 3 & & \\ & & & & & & & 2 & \\ & & & & & & & & 2 \end{bmatrix}$$

$J_1(4)$ $J(4)$ $J_2(4)$ $J(3)$ $J(2)$

$\lambda_1 = 4$

$\lambda_2 = 3$

$\lambda_3 = 2$

A.M $(4) = 5$ AM $(3) = 2$ A.M $(2) = 2$

G.M $(4) = 2$ G.M$(3) = 1$ G.M $(2) = 2$

- The Number of blocks in $J(\lambda_i)$ is equal to the G.M. of λ_i.

. . . .

$$A^k = P J^k P^{-1}$$

All $|\lambda_i| < 1$ except $\lambda_1 = 1$

$$\lim_{k\to\infty} J^k = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \; 0 \end{bmatrix} \qquad \lim_{k\to} J^k(\lambda_i) = 0$$

for $i = 2, 3 \ldots n$

For instan

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.1 & 1 \\ 0 & 0 & 0.1 \end{bmatrix} \xrightarrow{J^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.01 & 0.2 \\ 0 & 0 & 0.01 \end{bmatrix} \xrightarrow{J^3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.001 & 0.03 \\ 0 & 0 & 0.001 \end{bmatrix}$$

$$\downarrow J^4$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{J^7} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0.0001 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{J^6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0.005 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{J^5} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.0001 & 0.004 \\ 0 & 0 & 0.001 \end{bmatrix}$$

The sol^n of closed loop system

$$x(k) = A_N x(0)$$

$$x(k) = P J^k P^{-1} x(0) = P \begin{bmatrix} 1 & & & \\ & J^k(\lambda_2) & & \\ & & \ddots & \\ & & & J^k(\lambda_n) \end{bmatrix} P^{-1}$$

Since $|\lambda_i| < 1$ for $i = 2, 3 \cdots n$ (n - distinct eigenvalues of A_N)

$$J^k(\lambda_i) \xrightarrow{k\to\infty} 0$$

Hence as $k \to \infty$

$$J^k \longrightarrow \left[\begin{array}{c|ccc} 1 & & & \\ \hline & 0 & & \\ & & \ddots & \\ & & & 0 \end{array}\right]$$

Then the solution

$$x(k) = PJ^kP^{-1}x(0) \xrightarrow{k\to\infty} P \left[\begin{array}{c|ccc} 1 & & & \\ \hline & 0 & & \\ & & \ddots & \\ & & & 0 \end{array}\right] P^{-1}x(0)$$

Let $P = \begin{bmatrix} | & | & & | \\ p_1 & p_2 & \cdots & p_n \\ | & | & & | \end{bmatrix}$ $P^{-1} = \begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix}$

Then

$$x(k) \xrightarrow{k\to\infty} \begin{bmatrix} | & | & & | \\ p_1 & p_2 & \cdots & p_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \begin{bmatrix} q_1^T x(0) \\ q_2^T x(0) \\ \vdots \\ q_n^T x(0) \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ p_1 & p_2 & \cdots & p_n \\ | & & & | \end{bmatrix} \begin{bmatrix} q_1^T x_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = p_1 q_1^T x_0$$

p_1 & q_1 are the right & left eigenvectors, respectively of A_N corresponding to eigenvalue 1.

Here $p_1 = \mathbb{1}$

$$x(k) \xrightarrow{k \to \infty} (q_1^T x_0) \cdot \mathbb{1} \in \text{span}\{\mathbb{1}\}$$

Agreement subspace.

→ Primitive Matrices:

A non-negative matrix $M \in \mathbb{R}^{n \times n}$ is said to be "primitive" if it is irreducible and it has only one eigenvalue of maximum modulus.

- The normalized Adjacency matrix A_N is a primitive matrix.

A_N is irreducible

$\rho(A_N) = 1$ has A.M. $= 1$.

• **Result** [Horn & Johnson, Matrix Analysis book]

If M is non-negative and primitive, then

$$\boxed{\lim_{k \to \infty} \left[\frac{1}{r} M \right]^k = x y^T}$$ (xy^T is positive matrix)

when

$r = \rho(M)$ (spectral radius)

$Mx = rx$ (right eigenvector of M corresponding to r)

$M^T y = ry$ (left eigenvector of M corresponding to r)

$x > 0$ $(x_i > 0)$

$y > 0$ $(y_i > 0)$

and $x^T y = 1$

→ Note that A_H is primitive & non-negative

⇓ According to above result

$$\lim_{k \to \infty} A^k = p_1 q_1^T$$ where p_1 & q_1 are the left & right eigenvectors of A_H to e.v. 1.

$r = 1$

$$\Rightarrow \lim_{k \to \infty} x(k) = \lim_{k \to \infty} A_H^k x(0) = p_1 q_1^T x(0)$$

$$= (q_1^T x(0)) \cdot \mathbb{1}.$$