ELL 805

Lecture- 16

Gershgorin Disc Theorem

- For a given matrix $A \in \mathbb{R}^{n \times n}$, define the following discs:

$$D_i := \{ s \in \mathbb{C} : |s - c_i| \leq \rho_i \}$$

where $c_i = a_{ii}$ ← Center of the disc

$$\rho_i = \sum_{\substack{j=1,2..n \\ i \neq j}} |a_{ij}|$$ ← Radius of the disc.

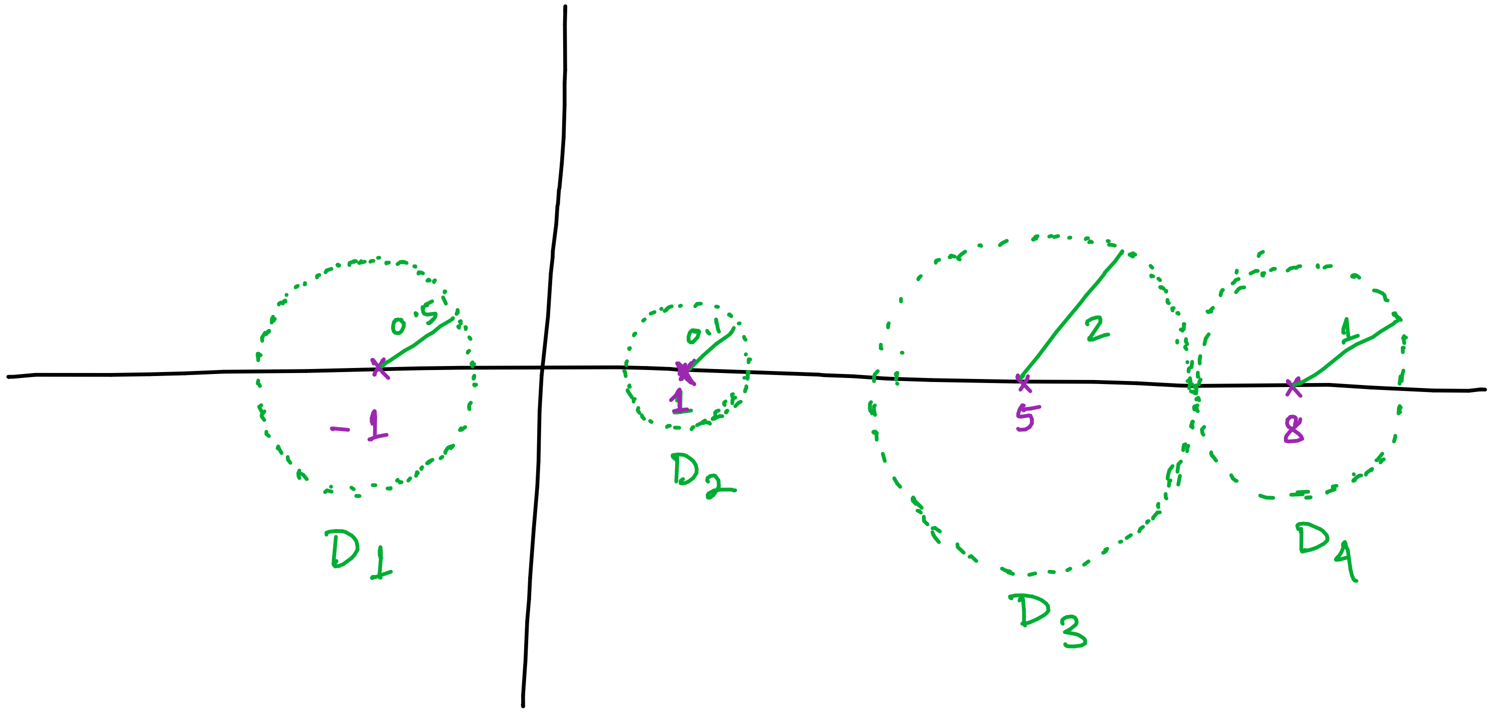
Then the result says:

The eigenvalues of A belong to the union of all D_i's. i.e $\bigcup_{i=1}^{n} D_i$.

$$A = \begin{bmatrix} -1 & 0.3 & 0 & -0.2 \\ 0 & 1 & 0.1 & 0 \\ 1 & -0.5 & 5 & 0.5 \\ 1 & 0 & 0 & 8 \end{bmatrix}$$

$D_1 \begin{cases} c_1 = -1 \\ \rho_1 = 0.5 \end{cases}$ $D_2 \begin{cases} c_2 = 1 \\ \rho_2 = 0.1 \end{cases}$

$D_4 \begin{cases} c_4 = 8 \\ \rho_4 = 1 \end{cases}$ $D_3 \begin{cases} c_3 = 5 \\ \rho_3 = 2 \end{cases}$



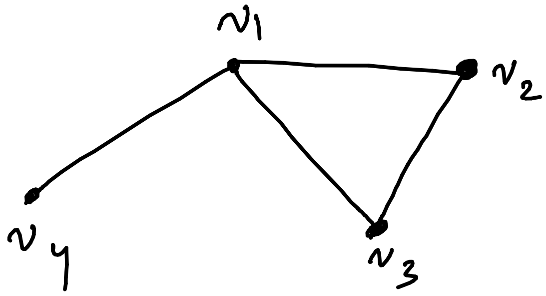
The eigenvalues of A are in $\bigcup_{i=1}^{4} D_i$.

→ Assume that the dynamics of the agents are:

$$x_i(k+1) = x_i(k) + u_i(k), \quad x(0) \text{ initial state}$$

$$\boxed{u_i(k) = \epsilon \left[\sum_{j \in \mathcal{N}(i)} \big(x_j(k) - x_i(k) \big) \right]}$$

ϵ is a small positive number & it is a design parameter.



$$u_1(k) = \epsilon \left[(x_3 - x_1) + (x_2 - x_1) + (x_4 - x_1) \right]$$

$$u_2(k) = \epsilon \left[(x_1 - x_2) + (x_3 - x_2) \right]$$

$$u_3(k) = \epsilon \left[(x_2 - x_3) + (x_1 - x_3) \right]$$

$$u_4(k) = \epsilon (x_1 - x_4)$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \epsilon \underbrace{\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}}_{-L} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

$\uparrow$ Laplacian matrix of n/w graph

$$u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_n(k) \end{bmatrix} = -\epsilon L \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}}_{x(k)}$$

$$x_i(k+1) = x_i(k) + u_i(k)$$

$$x(k+1) = x(k) + u(k)$$

$$= x(k) - \epsilon L\, x(k)$$

$$= \underbrace{(I - \epsilon L)}_{P}\, x(k)$$ $\leftarrow$ closed loop system dynamics.

$$P = I - \epsilon L = I - \epsilon(\Delta - A)$$

$\Delta \leftarrow$ Degree matrix

$A \leftarrow$ Adjacency mat

$$= (I - \epsilon \Delta) + \epsilon A$$

Choose $\boxed{\epsilon < \frac{1}{d_{max}}}$ when $d_{max} = \max(d(v_i))$

$\uparrow$ degree of vertex

For our example $d_{max} = 3$

$$\Rightarrow \quad \epsilon < \frac{1}{3} = 0.33$$

Let us chose $\epsilon = 0.2$.

$$\Delta = \begin{bmatrix} 3 & & & \\ & 2 & & \\ & & 2 & \\ & & & 1 \end{bmatrix}$$

$$P = (I - \epsilon\Delta) + \epsilon A$$

$$= \begin{bmatrix} 0.4 & & & \\ & 0.6 & & 0 \\ 0 & & 0.6 & \\ & & & 0.8 \end{bmatrix} + \begin{bmatrix} 0 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0 & 0.2 & 0 \\ 0.2 & 0.2 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 & 0 \\ 0.2 & 0.2 & 0.6 & 0 \\ 0.2 & 0 & 0 & 0.8 \end{bmatrix}$$

- P is a non-negative matrix
- P is a stochastic matrix (doubly)

$\swarrow$ spectral radius $\rho(P) = 1$

$\Downarrow$ $\left\{ \begin{array}{l} P \cdot \mathbb{1} = \mathbb{1} \\ \mathbb{1}^T P = \mathbb{1}^T \end{array} \right.$ $\rightarrow$ 1 is an eigenvalue of P & the corresponding eigenvector $\mathbb{1}$.

- The digraph associated with P is "strongly connected" iff the network graph is connected.

- Since we are assuming connected n/w topology

$\Downarrow$

the matrix P has "strongly connected" property.

$\Downarrow$

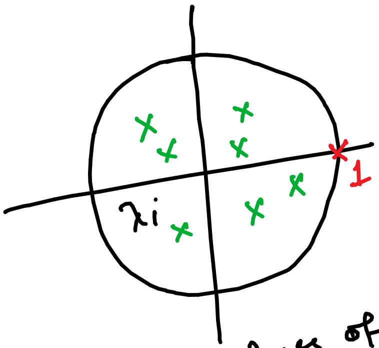
P is irreducible

- P is non-negative & irreducible matrix

$\Downarrow$ Perron-Frobenius Result

$\left\{ \begin{array}{l} \bullet\ \rho(P) = 1 \text{ is an eigenvalue of } P \\ \bullet\ \text{The eigenvalue } 1 \text{ has A.M. equal to one.} \end{array} \right.$

Except the eigenvalue $\lambda_1 = 1$ of P, the other eigenvalues are strictly inside the unit disc in Complex plane.



Eigenvalues of P

$$P\mathbb{1} = \mathbb{1}$$

$\Downarrow$

$\mathbb{1}$ is an eigenvector of P corresponding to the eigenvalue 1.

- Since P is symmetric, it is diagonalizable.

For the considered ex. the eigenvalues of P are $= 0.2, 0.4, 0.8, 1$.

Closed loop System

$$x(k+1) = P\,x(k)$$

so that $\qquad x(k) = P^k x(0)$

$$\boxed{P^k = \lambda_1^k v_1 w_1^T + \lambda_2^k v_2 w_2^T + \cdots + \lambda_n^k v_n w_n^T}$$

v_i & w_i are the right & left eigenvectors, respectively of P corresponding to eigenvalue λ_i.

$v_1 = \mathbb{1}$, $\lambda_1 = 1$

$|\lambda_i| < 1$ for $i = 2, 3 \ldots n$

$$x(k) \xrightarrow{k \to \infty} v_1 w_1^T x(0) = \underbrace{(w_1^T x(0)) \cdot \mathbb{1}}_{\in \text{ span}\{\mathbb{1}\}} .$$

$\in$ span $\{\mathbb{1}\}$

"

Agreement subspace

$\Downarrow$

Consensus achieved using the feedback control.

Use Gersgorin's th^m

$$P = \begin{bmatrix} 0.4 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 & 0 \\ 0.2 & 0.2 & 0.6 & 0 \\ 0.2 & 0 & 0 & 0.8 \end{bmatrix}$$

$c_1 = 0.4$
$r_1 = 0.6$

$c_2 = 0.6$

$r_2 = 0.4$

$c_3 = 0.6$

$r_3 = 0.4$

$c_4 = 0.8$

$r_4 = 0.2$

Center $c_i = 1 - \epsilon d(v_i)$

Radius $r_i = \epsilon d(v_i)$