ELL805

Lecture - 17

- The communication among the agents is bidirectional, which is represented as an undirected graph.

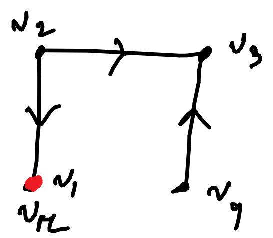
→ Unidirectional communication
$\downarrow$ Represented by
Directed Graphs (digraphs)

→ **Rooted Out-branching (ROB):**

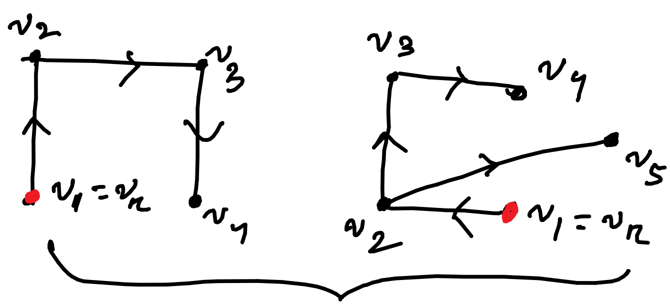
A digraph is rooted out-branching if it has the following properties:

(i) it does not contain a directed cycle,

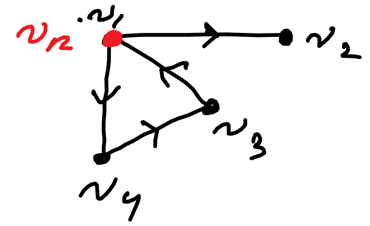
(ii) it has a vertex v_r (root-vertex) such that for every vertex v_i in the digraph, there is a directed path from v_r to v_i.



Not a ROB

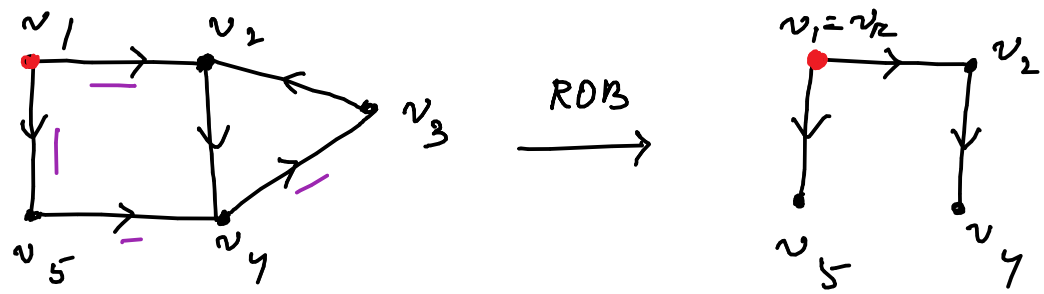


ROB



Not ROB

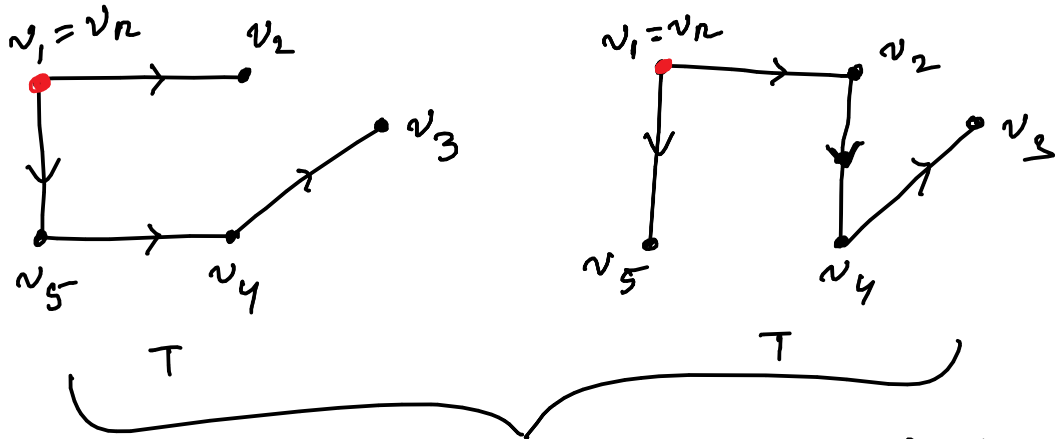
→ For a given digraph G_d, one can obtain a ROB of G_d, which is a subdigraph of G_d.



G_d

→ Spanning Rooted Out-branching (SROB):

• It is a subdigraph $T(G', v')$ of digraph $G_d(V, E)$, s.t. $\begin{cases} V' = V & \leftarrow \text{vertex set} \\ E' \subseteq E & \leftarrow \text{Edge set.} \end{cases}$



S.ROB corresponding to the root vertex $v_1 = v_r$

- Let a digraph G_d is given. The associated

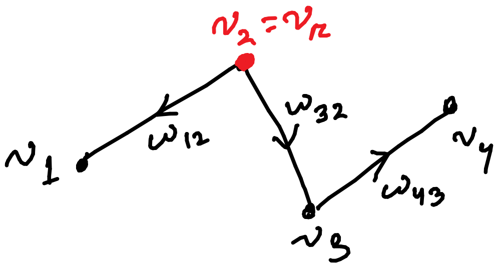
→ Adjacency matrix $A(G_d)$ &

→ Laplacian matrix $L(G_d)$

→ Let T_{v_i} be the set of all SROBs of G_d corresponding to the root vertex v_i. The elements of T_{v_i} are denoted as T, which is a SROB of G_d.

→ The product of edge weights that appear in a SROB 'T' be $\bar{\omega}(e)$ i.e.

$$\bar{\omega}(e) := \prod_{e_{ij} \in T} w(e_{ij})$$



T

$\rightarrow \quad \bar{\omega}(e) = w_{12} w_{32} \omega_{43}$

e_{ij} is an edge between vertex v_i & v_j

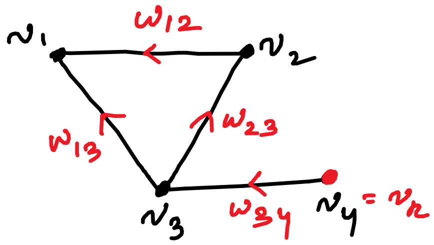
→ $L_{v_i}(G_d)$: The matrix that is obtained from $L(G_d)$ by deleting the ith row & ith column.

- Tutte's Matrix-Tree Theorem

Let $v_i = v_r$ be a root of SROB of a weighted digraph G_d. Then

$$\boxed{\det\left(L_{v_i}(G_d)\right) = \sum_{T \in T_{v_i}} \bar{\omega}(e)}$$

Example



In-degree vertex matrix

$$\Delta_{in} = \begin{bmatrix} w_{12}+w_{13} & & & \\ & w_{23} & & 0 \\ & 0 & w_{34} & \\ & & & 0 \end{bmatrix}$$

$$A(G_d) = \begin{bmatrix} 0 & w_{12} & w_{13} & 0 \\ 0 & 0 & w_{23} & 0 \\ 0 & 0 & 0 & w_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L(G_d) = \Delta_{in} - A(G_d)$$

$$= \begin{bmatrix} w_{12}+w_{13} & -w_{12} & -w_{13} & 0 \\ 0 & w_{23} & -w_{23} & 0 \\ 0 & 0 & w_{34} & -w_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

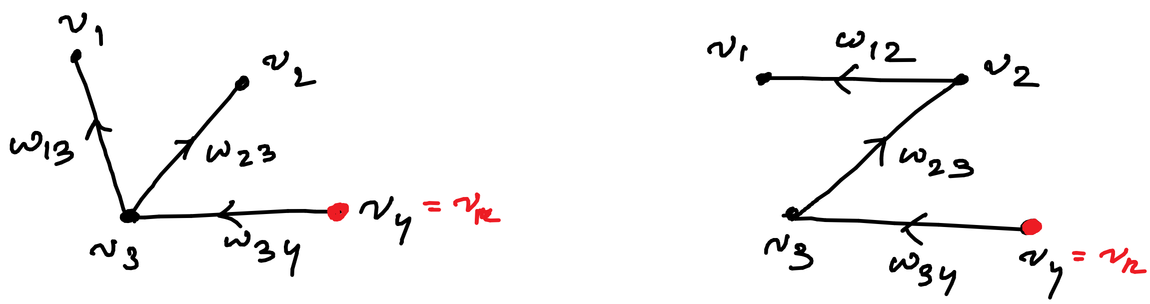
- We want to compute $\det\left(L_{v_4}(G_d)\right)$

obtained by deleting 4^{th} row & 4^{th} column of $L(G_d)$.

→ Constructing the set T_{v_4}

↓

The set of all SRDB corresponding to the vertex root v_4.



T $\qquad$ T

T_{v_4}

$$\bar{w}(e) = w_{13}w_{23}w_{34} \qquad \bar{w}(e) = w_{12}w_{23}w_{34}$$

$$+ \qquad \sum_{T \in T_{v_4}} \bar{w}(e)$$

$$\downarrow$$

$$(w_{12} + w_{13})\, w_{23} w_{34}$$

$$\parallel$$

$$\det\left[L_{v_4}(G_d)\right]$$

→ Observe that none of the other vertices in G_d can be considered as a root vertex. Hence,

$$\det\left(L_{v_1}(G_d)\right) = \det\left(L_{v_2}(G_d)\right) = \det\left(L_{v_3}(G_d)\right) = 0$$

→ Result

A digraph G_d on 'n' vertices contains a SROB as a subdigraph if and only if $\text{rank}(L(G_d)) = n-1$. In that case $\mathcal{N}(L(G_d)) = \text{span}\{\mathbb{1}\}$.

Proof

Let the characteristic polynomial of $L(G_d)$ be

$$\alpha(s) = s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

Since row sums are zero, $L(G_d)$ is a singular matrix. ⇒ at least one eigenvalue of $L(G_d)$ is 0.

$$\alpha_0 = \prod_i \lambda_i = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_n = 0$$

λ_i : an eigenvalue of $L(G_d)$.

[• The rank of a matrix A is equal to the number of non-zero eigenvalues of A.]

$$\Rightarrow \quad \text{rank}(L(G_d)) \leq n-1$$

Let the vertex set of G_d be $\mathcal{V} := \{v_1, v_2 \cdots v_n\}$

The coefficient α_1 can be computed as follows:

$$\alpha_1 = \sum_{v_i \in V} \det\left(L_{v_i}(G_d)\right)$$ ← Obtain the SRDBs of G_d by considering all vertices v_i as the root vertex of G_d.

By using Tutte's Matrix-Tree result,

$\det\left(L_{v_i}(G_d)\right) \neq 0$ if and only if G_d has a SROB.

$$\Updownarrow$$

$$\alpha_1 \neq 0$$

$$\Updownarrow$$

$L(G_d)$ has only one zero eigenvalue.

(If $\alpha_1 = 0 \Rightarrow \alpha(s) = s^2(s^{n-2} + \alpha_{n-1} s^{n-3} + \cdots + \alpha_3)$
& hence there will be two roots of $\alpha(s)$ at 0)

Here, $\operatorname{rank}\left[L(G_d)\right] = n-1$ iff G_d has a SROB.

Note that $L(G_d)$ has now known zero

Hu $L(G_d) \cdot \mathbb{1} = 0$

Hence $\qquad \mathbb{1} \in \mathcal{N}\left(L(G_d)\right)$

null space

The geometric multiplicity of an eigenvalue λ_i is less or equal to the algebraic multiplicity of λ_i.
(AM)

Since the eigenvalue $\lambda_n = 0$ of $L(G_d)$ has AM $= 1$, the G.M. of $\lambda_n = 1$.

Hence $\quad \dim\left(\mathcal{N}\left(L(G_d)\right)\right) = 1$

Hence $\quad \mathcal{N}\left(L(G_d)\right)$ is : $\text{span}\{\mathbb{1}\}$.