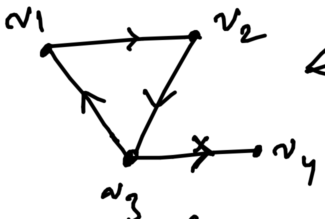
ELL805

Lecture - 18



G_d

$\leftarrow$ $\exists$ one SROB

$\Downarrow$

$$\text{rank}\left(L(G_d)\right) = n-1$$

$\uparrow$ In-degree Laplacian matrix.

$\rightarrow$ Consider the agent dynamics

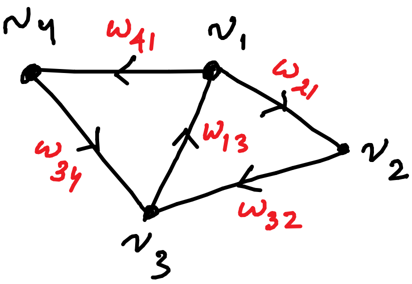
$$\dot{x}_i = u_i \qquad x_i(0) = x_{i_0}$$

Feedback control :

$$u_i = \sum_{j \in \mathcal{N}_I(i)} w_{ij} (x_j - x_i)$$

The set $\mathcal{N}_I(i) = \{ j :$ there is an edge in G_d from v_j to $v_i \}$

$\nearrow$ In-degree neighborhood set



$$\mathcal{N}_I(\underset{i}{1}) = \{3\} \qquad \mathcal{N}_I(3) = \{2,4\}$$

$$\mathcal{N}_I(2) = \{1\} \qquad \mathcal{N}_I(4) = \{1\}$$

$$u_1 = w_{13}(x_3 - x_1) \qquad u_4 = w_{41}(x_1 - x_4)$$

$$u_2 = w_{21}(x_1 - x_2)$$

$$u_3 = w_{32}(x_2 - x_3) + w_{34}(x_4 - x_3)$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -w_{13} & 0 & w_{13} & 0 \\ w_{21} & -w_{21} & 0 & 0 \\ 0 & w_{32} & -(w_{32}+w_{34}) & w_{34} \\ w_{41} & 0 & 0 & -w_{41} \end{bmatrix}}_{-\mathcal{L}(G_d)} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

- Not symmetric
- Column sum is zero
  $\mathcal{L}(G_d)\mathbb{1} = 0$

$-\mathcal{L}(G_d)$ in-degree Laplacian matrix
$\;\;\;\;\;\; \| $
$(\Delta_{in} - A)$

$$u = -\mathcal{L}(G_d)\,x \qquad x_i \in \mathbb{R}$$

→ Closed loop system:

$$\dot{x} = -\mathcal{L}(G_d)\,x \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

→ Where are the eigenvalues of $-\mathcal{L}(G_d)$?

↓ Apply Gersgorin Disc result

The eigenvalues are contained in the union of following discs:

$$D_i := \{ z \in \mathbb{C} \;:\; |z - d_{in}| \leq r_{in} \}$$

r_{in} : the radius of D_i

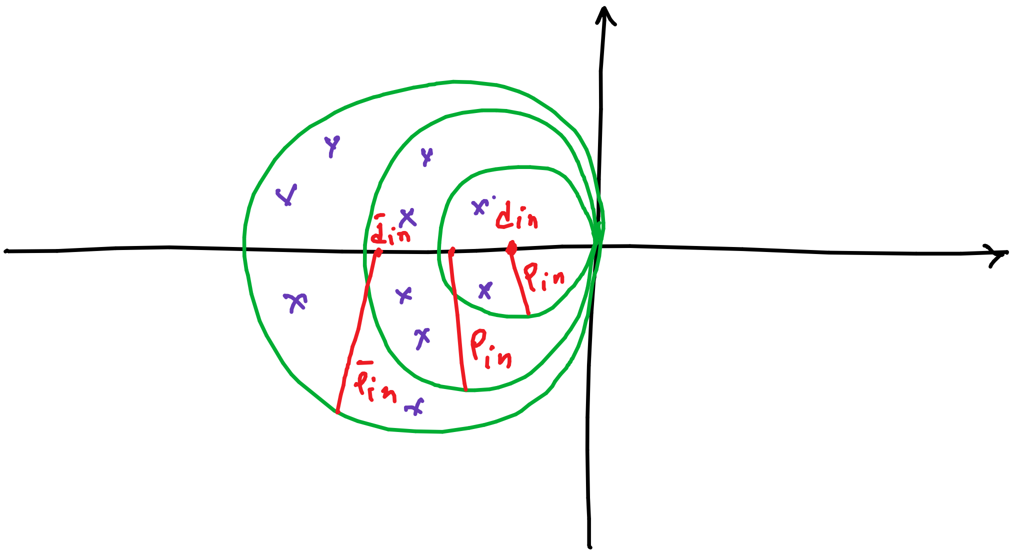
d_{in} : Center of D_i

For the considered in-degree Laplacian matrix, centers are

$$d_{in} = - \sum_{j \in N_I(i)} w_{ij}$$

$$(w_{ij} > 0)$$

$$\rho_{in} = \sum_{j \in N_I(i)} w_{ij}$$



The eigenvalues of $-L(G_d)$ are contained in $\bigcup_{i=1}^{K} D_i$.

Disc

$$D_m := \{ z \in \mathbb{C} : |z - \bar{d}_{in}| \leq \bar{\rho}_{in} \}$$

$\bar{d}_{in}$ = maximum in-degree, $\bar{\rho}_{in} = |\bar{d}_{in}|$.

- Since all the discs contained in D_m, the eigenvalues of $-L(G_d)$ contains in D_m.

Assume that $-\mathcal{L}(G_d)$ has k number of distinct eigenvalues.

$$\Downarrow$$

$\exists$ a non-singular matrix P s.t.

$$-P^{-1}\mathcal{L}(G_d)P = -J = \begin{bmatrix} -J(\lambda_1) & & & 0 \\ & -J(\lambda_2) & & \\ & & \ddots & \\ 0 & & & -J(\lambda_k) \end{bmatrix}$$

where

$$J(\lambda_i) = \begin{bmatrix} J_1(\lambda_i) & & & 0 \\ & J_2(\lambda_i) & & \\ & & \ddots & \\ 0 & & & J_s(\lambda_i) \end{bmatrix}$$

$$J_*(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}_{\beta_i \times \beta_i} = \lambda_i I + \underbrace{\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}}_{\text{Nilpotent matrix } N_{\beta_i}}$$

$$J_*(\lambda_i) = \lambda_i I_{\beta_i} + N_{\beta_i}$$

$$e^{Jt} = I + \sum_{g=1}^{\infty} \frac{t^g J^g}{g!}$$ (Power series representation)

$$e^{J_*(\lambda_i)t} = e^{(\lambda_i I_{\beta_i} + N_{\beta_i})t}$$

$$= e^{\lambda_i I_\beta t} \cdot e^{N_{\beta_i} t} \quad (\text{Since } I_\beta \text{ \& } N_{\beta_i} \text{ commute})$$

$$= e^{\lambda_i t} \underbrace{\begin{bmatrix} 1 & t & \cdots & \frac{t^{\beta_i - 1}}{(\beta_i - 1)!} \\ 0 & 1 & \cdots & \frac{t^{\beta_i - 2}}{(\beta_i - 2)!} \\ & & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}}_{H(t)}$$

→ Assuming that the reference graph contains a SROB $\Rightarrow$

$$\begin{cases} \text{rank}(L(G_d)) = n - 1 \\ L(G_d) \text{ has only } \underline{\text{one}} \text{ eigenvalue at '0'} \\ \dim(\mathcal{N}(L(G_d))) = 1 \end{cases}$$

$$L(G_d) \cdot \mathbb{1} = 0$$

Further, non-zero eigenvalues of $-L(G_d)$ are in the open left half of $\mathbb{C}$.

$$-P^{-1}L(G_d)P = -J = \left[\begin{array}{cccc|c} -J(\lambda_1) & & & & \\ & -J(\lambda_2) & & 0 & \\ & & \ddots & & \\ & 0 & & -J(\lambda_{n-1}) & \\ \hline & & & & 0 \end{array}\right]$$

$$\lim_{t\to\infty} e^{-J_*(\lambda_i)t} = \lim_{t\to\infty} e^{-\lambda_i t} H(t) \qquad (\lambda_i > 0)$$

$$= 0$$

$$\lim_{t\to\infty} e^{-Jt} = \lim_{t\to\infty} \left[\begin{array}{cccc|c} e^{-J(\lambda_1)t} & & & 0 & \\ & e^{-J(\lambda_2)t} & & & \\ & & \ddots & & \\ & 0 & & e^{-J(\lambda_{n-1})t} & \\ \hline & & & & e^{0} \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ \hline & & & & 1 \end{array}\right] \quad \cdots \cdots (*)$$

$$\dot{x} = -L(G_d)x$$

$$\Rightarrow x(t) = e^{-L(G_d)t}x_0$$

$$-P^{-1}L(G_c)P = J \quad \Rightarrow \quad L(G_c) = -PJP^{-1}$$

Then

$$x(t) = e^{-L(G_c)t}x_0 = e^{-PJP^{-1}t}x_0$$

$$= P e^{-Jt} P^{-1} x_0$$

Show this

$$\Rightarrow \lim_{t\to\infty} x(t) = \lim_{t\to\infty} P e^{-Jt} P^{-1} x_0$$

$$= \lim_{t\to\infty} P \left[\begin{array}{cccc|c} e^{-J(\lambda_1)t} \to 0 & & & & \\ & e^{-J(\lambda_2)t} \to 0 & & & 0 \\ & 0 & \ddots & & \\ & & & e^{-J(\lambda_{n-1})t} \to 0 & \\ \hline & & 0 & & e^0 \end{array}\right] P^{-1} x_0$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ p_1 & p_2 & \cdots & p_n \\ | & | & & | \end{bmatrix}}_{P} \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 1 \end{bmatrix} \underbrace{\begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix}}_{P^{-1} = Q} x_0$$

$$= p_n q_n^T x_0$$

$$\lim_{t \to \infty} x(t) = x^* = (q_n^T x_0) p_n$$

where p_n & q_n^T are the right and left eigenvectors of $-L(G_d)$ corresponding to the eigenvalue $\lambda_n = 0$.

Recalls that $L(G_d) \cdot \mathbb{1} = 0$ (column sum 0)

$\rightarrow$ the eigenvector $p_n = \text{span}\{\mathbb{1}\}$.

However, the row sum may not be '0'.

Hence q_n^T may not belong to $\text{span}\{\mathbb{1}\}$.

- Hence in a directed communication topology, the consensus can be achieved for the agents: $\dot{x}_i = u_i$ with $u_i = \sum_{j \in N_G(i)} w_{ij} (x_j - x_i)$ if and only if the associated directed graph contains a SRDB. The consensus point $\boxed{x^* = (q_n^T x_0) p_n}$.