ELL805

Lecture - 19

$$\dot{x}_i = u_i$$

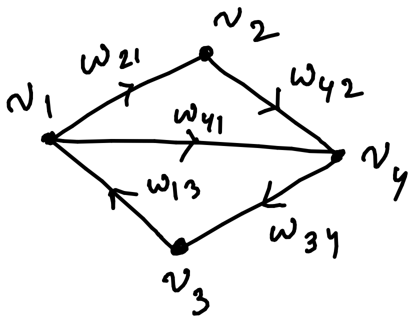
with $\quad u_i = \sum_{j \in \mathcal{N}_g(i)} w_{ij} (x_j - x_i)$

Consensus can be achieved among x_i if and only if the network digraph contains a SRDB.

$$\lim_{t \to \infty} x(t) = x^* = (q_n^T x_0) \cdot p_n$$

$p_n \in \text{span}\{\mathbb{1}\}$

May not be average consensus.



Balanced Digraph (Lecture - 7)

- In-degree = Out degree at every vertex

- For a balanced digraph, when the edge weights are 1, at each vertex, the number of incoming edges = outgoing edges

$$d_{in}(v_1) = w_{13} = d_o(v_1) = w_{21} + w_{41}$$

$$d_{in}(v_2) = w_{21} = d_o(v_2) = w_{42}$$

$$d_{in}(v_3) = w_{34} = d_o(v_3) = w_{13}$$

$$d_{in}(v_4) = w_{42} + w_{41} = d_{out}(v_4) = w_{34}$$

$$\Delta_{in} = \begin{bmatrix} w_{13} & & & \\ & w_{21} & & \\ & & w_{34} & \\ & & & w_{42}+w_{41} \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & w_{13} & 0 \\ w_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & w_{34} \\ w_{41} & w_{42} & 0 & 0 \end{bmatrix}$$

In-degree Laplacian matrix

$$\mathcal{L}(G_d) = \begin{bmatrix} w_{13} & 0 & -w_{13} & 0 \\ -w_{21} & w_{21} & 0 & 0 \\ 0 & 0 & w_{34} & -w_{34} \\ -w_{41} & -w_{42} & 0 & w_{42}+w_{41} \end{bmatrix} = 0$$

For a balanced digraph $\rightarrow \begin{cases} \mathcal{L}(G_d)\cdot \mathbb{1} = 0 \\ \mathbb{1}^T \mathcal{L}(G_d) = 0 \end{cases}$

$\Downarrow$

- Considering the previous discussion, for a closed loop system $\dot{x} = -\mathcal{L}(G_d)\,x$ with digraph G_d satisfying: $\begin{cases} \bullet \text{ contains a SROB} \\ \bullet \text{ it is balanced} \end{cases}$

$\downarrow$ convergence

$$\lim_{t\to\infty} x(t) = x^* = (q_n^T x_0)\cdot p_n = \left(\frac{\mathbb{1}^T x_0}{n}\right)\cdot \mathbb{1}$$

Since $q_n, p_n \in \text{span}\{\mathbb{1}\}$

- A weakly connected balanced digraph is strongly connected.

→ **Result**

The closed loop system $\dot{x} = -L(G_d)x$ with G_d weakly connected & balanced reaches to the average consensus from every initial condition.

Proof

Since G_d is weakly connected & balanced

$\Downarrow$

G_d is strongly connected

$\Downarrow$

G_d contains a SROB

$\Downarrow$

Consensus can be achieved.

Further, since the digraph is balanced

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} \left[e^{-L(G_d)t} x_0 \right] = \left(\frac{\mathbb{1}^T x_0}{n} \right) \cdot \mathbb{1}.$$

average consensus

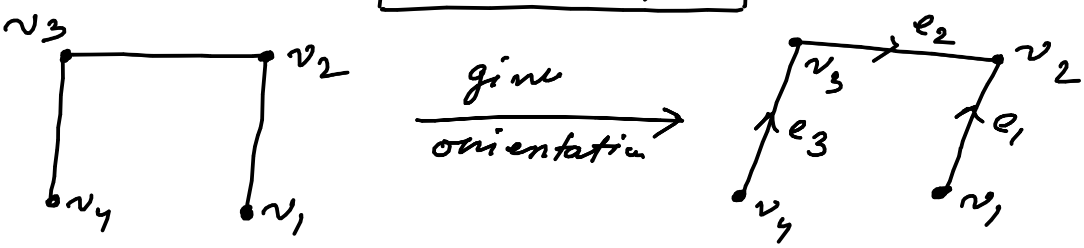
$\blacksquare$

→ Edge Agreement Protocol (Undirected Graph)

↑ Another approach to see convergence to consensus in undirected graph.

→ Consider the closed loop system (agreement protocol)

$$\boxed{\dot{x} = -L(G)\,x} \quad \cdots\cdots (*)$$



Incidence matrix

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad D^T = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\delta := D^T x = \begin{bmatrix} x_2 - x_1 \\ x_2 - x_3 \\ x_3 - x_4 \end{bmatrix}$$

$$\dot{\delta} = D^T \dot{x}$$

$$= -D^T L(G)\,x$$

$$= -D^T D \underline{D^T x}$$

$$= -L_e(G)\,\delta$$

$L_e(G)$ ↑ Edge Laplacian matrix

$$\boxed{\dot{\delta} = -L_e(G)\,\delta}$$ ← Error dynamics on Edge Agreement Protocol

For consensus

↓ $t \to \infty$

$x_1 = x_2 = x_3 = \cdots = x_n$

⫴

$\delta \to 0$ as $t \to \infty$

→ Some Properties of Edge Laplacian Matrix :

- Let λ_i be a non-zero eigenvalue of Laplacian matrix L.

$$L v_i = \lambda_i v_i$$ where v_i is an eigenvector corresponding to eigenvalue λ_i

$$\Rightarrow DD^T v_i = \lambda_i v_i$$

$$\Rightarrow D^T D D^T v_i = \lambda_i D^T v_i$$

Let $\underbrace{x_i = D^T v_i}_{\text{Eigenvector relation}}$

$$\Rightarrow L_e x_i = \lambda_i x_i$$

- The non-zero eigenvalues of L an L_e are equal, however, the eigenvectors are different.
  ↳ The other eigenvalues of L_e are zero.

→ The null space of $L_e(G)$ = the null space of D.

show this

$$\mathcal{N}(L_e(G)) = \mathcal{N}(D)$$

Incidence matrix.

⇒ Let a vector $q \in \mathcal{N}(D)$

$$\Rightarrow Dq = 0 \equiv L_e q = 0$$

The eigenvectors of L_e are stacked in a matrix P

$$P = \begin{bmatrix} | & | & & | \\ p_1 & p_2 & \cdots & p_m \\ | & | & & | \end{bmatrix} \qquad P^T = Q = \begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ & \vdots & \\ - & q_m^T & - \end{bmatrix}$$

m: No of edges of G = the size of $L_e \in \mathbb{R}^{m \times m}$

L_e is symmetric $\Rightarrow$ P is orthogonal

Let L_e has $k = n-1$ no. of non-zero eigenvalues. (Graph G has n-vertices)
connected.

$$\underbrace{p_1 \; p_2 \; \cdots \; p_k}_{\substack{\text{Right eigenvectors} \\ \text{corresponding} \\ \text{to non-zero} \\ \text{eigenvalues of} \\ L_e}} \quad \underbrace{p_{k+1} \; p_{k+2} \; \cdots \; p_m}_{\substack{\text{Right eigenvectors} \\ \text{corresponding to} \\ \text{zero eigenvalues} \\ \text{of } L_e}}$$

$$\left.\begin{matrix} q_1^T \\ q_2^T \\ \vdots \\ q_k^T \end{matrix}\right\} \rightarrow \text{Left eigenvectors of } L_e \text{ corresponding to non-zero eigenvalues.}$$

$$\left.\begin{matrix} q_{k+1}^T \\ q_{k+2}^T \\ \vdots \\ q_m^T \end{matrix}\right\} \rightarrow \text{Left eigenvectors corresponding to } 0\text{-eigenvalues of } L_e.$$

Edge agreement Protocol

$$\dot{\delta} = -\mathcal{L}_e \delta \qquad \delta(0) = D^T x(0)$$

SVT^n

$$\delta = e^{-\lambda_1 t}(q_1^T \delta_0) p_1 + e^{-\lambda_2 t}(q_2^T \delta_0) p_2 + \cdots + e^{-\lambda_k t}(q_k^T \delta_0) p_k + \cdots + e^{-\lambda_m t}(q_m^T \delta_0) p_m$$

$$\left.\begin{array}{l} e^{-\lambda_1 t}(q_1^T \delta_0) p_1 \\ e^{-\lambda_2 t}(q_2^T \delta_0) p_2 \\ \vdots \\ e^{-\lambda_k t}(q_k^T \delta_0) p_k \end{array}\right\} \xrightarrow{t \to \infty} 0$$

$\lambda_1 \cdots \lambda_k$ are non-zero eigenvalues of $\mathcal{L}$.

$$\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_m = 0$$

$$\left.\begin{array}{l} e^{-\lambda_{k+1} t}(q_{k+1}^T \delta_0) p_{k+1} \\ \vdots \\ e^{-\lambda_m t}(q_m^T \delta_0) p_m \end{array}\right\} \xrightarrow{t \to \infty} \begin{array}{l} (q_{k+1}^T \delta_0) p_{k+1} \\ (q_{k+2}^T \delta_0) p_{k+2} \quad \cdots \quad (*) \\ \vdots \\ (q_m^T \delta_0) p_m \end{array}$$

$$q_{k+1}^T \mathcal{L}_e = \mathcal{L}_e q_{k+1} = 0 \xrightarrow{\mathcal{N}(\mathcal{L}_e) = \mathcal{N}(D)} D q_{k+1} = 0$$

($\mathcal{L}_e$ is symmetric)

or

$$q_{k+1}^T D^T = 0$$

$$\delta_0 = D^T x_0 \qquad q_{k+1}^T \delta_0 = q_{k+1}^T D^T x_0$$

$$= 0$$

For all $j = k+1, k+2, \cdots, m$

we have $q_j^T \delta_0 = q_j^T D^T x_0 = 0$

$$\Downarrow$$

$$\delta(t) \xrightarrow{t \to \infty} 0 \equiv x_i - x_j \xrightarrow{t \to \infty} 0$$

Hence consensus is achieved.