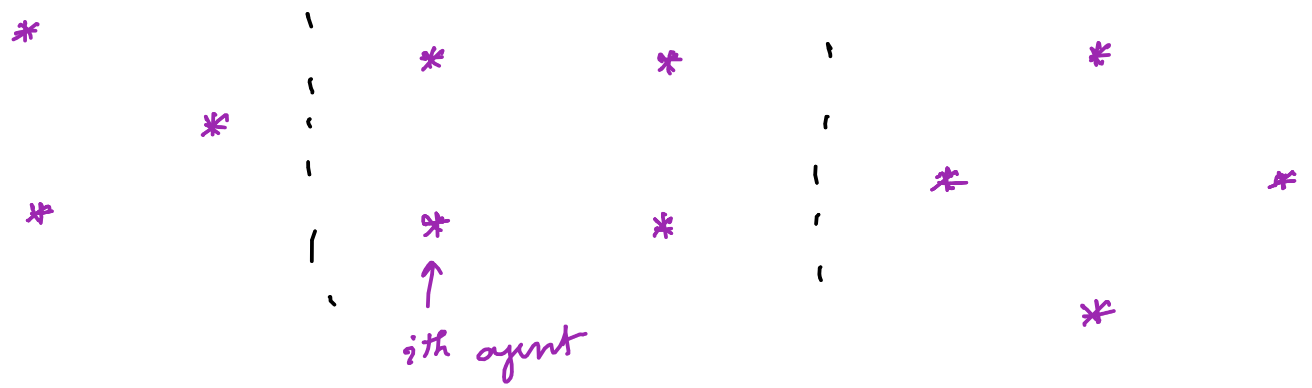
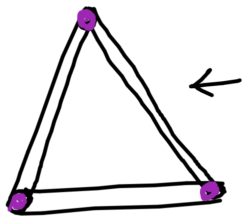
ELL 805

Lecture-20

Formation

↳ To realize a geometrical pattern by a set of agents (subsystems)





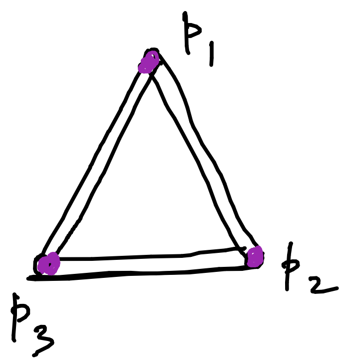
← bar-joint structure

bar → Rigid

joints → flexible

Each joint represents the agent's position (x_i or p_i)

Each bar represents the distance among the agents.



$x_i, p_i \in \mathbb{R}^n$

For mobile robots

$x_i, p_i \in \mathbb{R}^2$

For quadcopters $p_i \in \mathbb{R}^3$

Distance among the agents

$$d_{ij} = \|p_i - p_j\|_2 \text{ or } \|p_j - p_i\|_2$$

Formation Specifications

↳ Directly given the positions of the agents

↳ Specified in terms of distance among the agents

↳ Relative state specification.

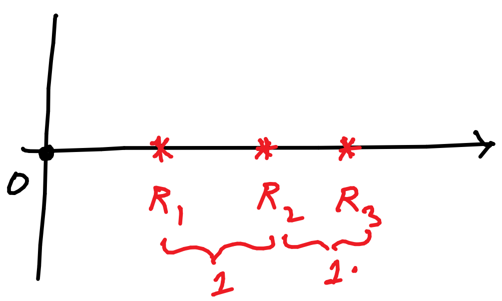
→ Define a set of distances

$$D := \left\{ \underset{\substack{\uparrow \\ \text{distance among agent-}i \text{ \& agent-}j}}{d_{ij}} \in \mathbb{R}^+ : d_{ij} = d_{ji},\ i \neq j \text{ for all } i, j = 1, 2 \cdots n \right\}$$

→ Let the 3-robots are allowed to move in a line.

A set of distance specifications $\left\{ \begin{array}{l} d_{12} = 1 \\ d_{23} = 1 \\ d_{13} = 2 \end{array} \right.$

$\Downarrow$

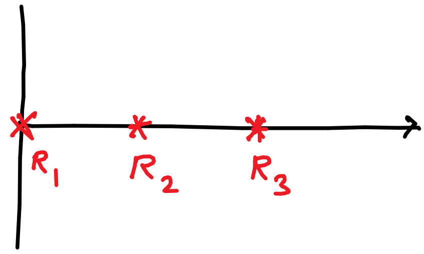


A set of position of three robots could be

$p_1 = 1 \quad p_2 = 2 \quad p_3 = 3$

→ Another distance specification for the three robots system

$$\left.\begin{aligned} d_{12} &= 1 \\ d_{23} &= 1 \\ d_{13} &= 3 \end{aligned}\right\}$$



↳ Can not be realized.

↓

- Instead of distance specification, we will say

"A feasible distance specification"

↓

There exists a solution for these set of equations

$$\left\{\begin{aligned} \|p_1 - p_2\| &= d_{12} \\ \|p_2 - p_3\| &= d_{13} \\ &\vdots \\ \|p_i - p_j\| &= d_{ij} \end{aligned}\right.$$

→ Scale - invariant Formation

Assume that the initial position of the agents are $p_1, p_2 \,..\, p_n$ and they satisfy: $\|p_i - p_j\| = d_{ij}$

(n - no. of agents)

Assume that the agents have moved to a new set of positions $x_1, x_2 \cdots x_n$

$p_1 \longrightarrow x_1$

$p_2 \longrightarrow x_2$

$\vdots$

$\underbrace{p_n}_{\substack{\text{Initial} \\ \text{position} \\ x_i(0)=p_i}} \longrightarrow \underbrace{x_n}_{\substack{\text{New} \\ \text{positions}}}$

We say that the formation is "scale-invariant" if

$$\boxed{\|x_i - x_j\| = \alpha d_{ij}}$$

where $\alpha > 0$ is a scaling factor.

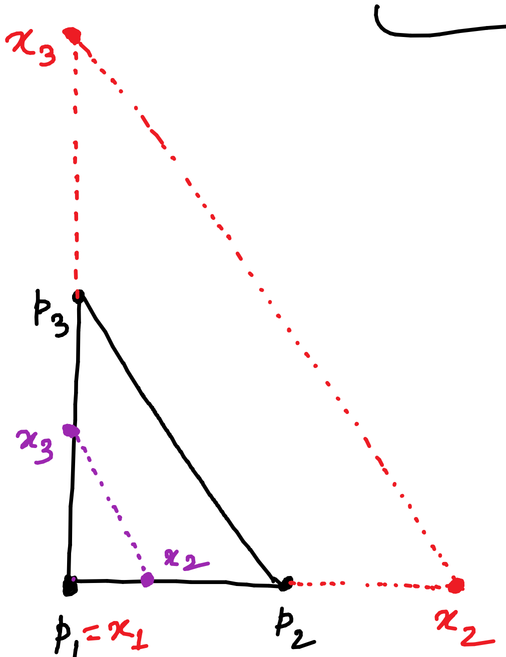
$\rightarrow$ Consider three agents, placed in a right-angle triangle. The initial positions are

$$\underbrace{p_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad p_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad p_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$\Rightarrow d_{12} = 1$

$d_{13} = 2$

$d_{23} = \sqrt{5}$



$\uparrow$

Scale-invariant formations

For $x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_2 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$, $x_3 = \begin{bmatrix} 0 \\ 0.4 \end{bmatrix}$

$\alpha = 0.2$

Let a set of new positions of agents be

$$\underbrace{x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}}$$

$\Rightarrow \|x_1 - x_2\| = 3 = 3 \cdot d_{12}$

$\|x_1 - x_3\| = 6 = 3 \cdot d_{13}$

$\|x_2 - x_3\| = 3\sqrt{5} = 3 d_{23}$

$\Rightarrow \alpha = 3$

→ Translation Invariant Formation

We are given with initial points

$p_1, p_2 \cdots p_n$ & they satisfy

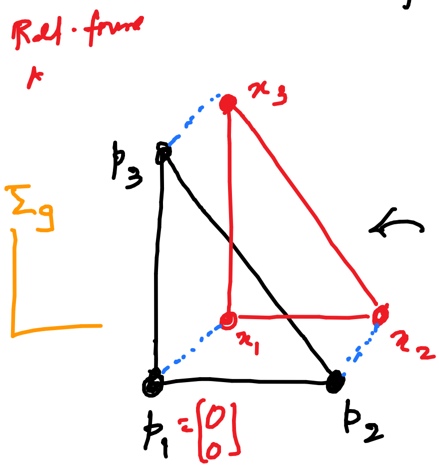
$\|p_i - p_j\| = d_{ij}$

Let the new positions of the agents are

$\boxed{x_i = p_i + \beta}$ $\quad \beta \in \mathbb{R}^n \quad x_i, p_i \in \mathbb{R}^n$

$x_1 = p_1 + \beta$
$x_2 = p_2 + \beta$
$\vdots$
$x_n = p_n + \beta$

$\|x_i - x_j\| = \|p_i + \beta - p_j - \beta\| = d_{ij}$



← Translation of the original formation in the direction of β

$p_1 = \begin{bmatrix}0\\0\end{bmatrix} \quad p_2 = \begin{bmatrix}1\\0\end{bmatrix} \quad p_3 = \begin{bmatrix}0\\2\end{bmatrix}$

Let $\beta = \begin{bmatrix}1\\1\end{bmatrix}$, Then :

$x_1 = \begin{bmatrix}1\\1\end{bmatrix} \quad x_2 = \begin{bmatrix}2\\1\end{bmatrix} \quad x_3 = \begin{bmatrix}1\\3\end{bmatrix}$

$\beta \rightarrow$ Determines the direct^n of motion.

Σ_g : Global co-ordinate frame

→ Rotation Invariant Formation

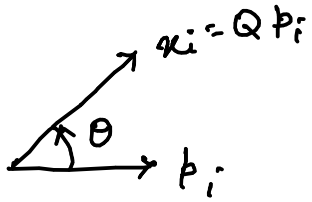
Given points $p_1, p_2 \cdots p_n$ satisfy $\|p_i - p_j\| = d_{ij}$

Define the new points for the agents

as $\boxed{x_i = Q p_i}$

where $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Rotation matrix



$$\|x_i - x_j\|_2^2 = \|Q p_i - Q p_j\|_2^2 \qquad \|k\|_2^2 = k^T k$$

$$= [Q(p_i - p_j)]^T [Q(p_i - p_j)]$$

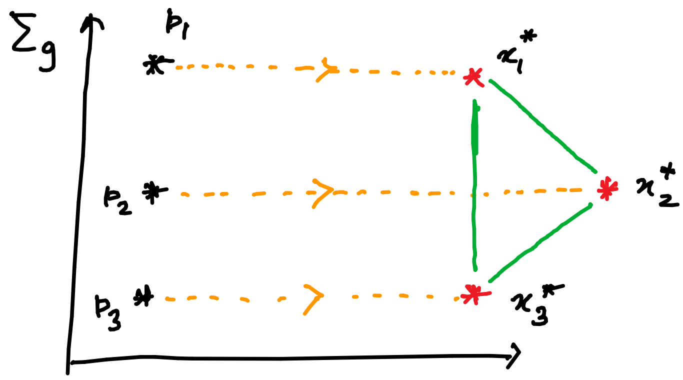
$$= (p_i - p_j)^T Q^T Q (p_i - p_j)$$

$$= \|p_i - p_j\|_2^2$$

$$= d_{ij}^2$$

→ How to achieve formation

→ Position-based Control



Σ_g is global co-ordinate system

$p_i = x_i(0)$

x_i → instantaneous position

x_i^* → final position of the agent-i

x_i^* are specified positions for desired formation.

objective : $x_i \longrightarrow x_i^*$ as $t \to \infty$

→ Let the agent dynamics be

$$\dot{x}_i = u_i \qquad x_i \in \mathbb{R}^n, \quad u_i \in \mathbb{R}^n$$

Let the feedback control for each agent be

$$u_i = k_p(x_i^* - x_i)$$

k_p is a positive scalar gain

closed loop system

$$\dot{x} = k_p(x_i^* - x_i)$$

Define an error vector $\delta_i = x_i^* - x_i$

$$\underbrace{\begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{bmatrix}}_{\delta} = \underbrace{\begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{bmatrix}}_{x^*} - \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x}$$

n-agents

$$\dot{\delta} = -\dot{x}$$

$$= -k_p(x_i^* - x)$$

$$= -k_p \delta$$

$$\delta(t) = e^{-k_p t}\delta(0)$$

Since $k_p > 0$ $\delta(t) \longrightarrow 0$ as $t \to \infty$

$$\Rightarrow x_i \longrightarrow x_i^*$$

→ For choosing different gains for individual agents $u_i = k_{p_i}(x_i^* - x_i)$

The error dynamics:

$$\begin{bmatrix} \dot{\delta}_1 \\ \dot{\delta}_2 \\ \vdots \\ \dot{\delta}_n \end{bmatrix} = \begin{bmatrix} -k_{p_1} I & & & \\ & -k_{p_2} I & & \\ & & \ddots & \\ & & & -k_{p_n} I \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{bmatrix}$$

Soln for $\delta_i(t) = e^{-k_{p_i} t}\delta_i(0)$

One can adjust different gains k_{p_i} for different agents.

In position-based control

- Agents need to sense their own position with respect to some global reference framework Σ_g.
- There is no need of inter-agent interaction.
- One needs only proportional control/ position control to achieve the desired formation.

$$x_i^* - x_i \qquad x_i - x_j = x_i^* - x_j^*$$

→ Kronecker Product

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} \qquad A \otimes B \neq B \otimes A$$

$$B \otimes A = \begin{bmatrix} b_{11}A & b_{12}A & b_{13}A \\ b_{21}A & b_{22}A & b_{23}A \end{bmatrix}$$

If $A \in \mathbb{R}^{n \times m}$ $B \in \mathbb{R}^{p \times q}$ $\rightarrow$ $A \otimes B \in \mathbb{R}^{np \times mq}$

- Properties

(i) $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$

(ii) $(\alpha A) \otimes B = \alpha (A \otimes B) = A \otimes (\alpha B)$

$\alpha \in \mathbb{R}$

(iii) $(A \otimes B)^T = A^T \otimes B^T$

(iv) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ provided A & B are non-singular.