ELL805

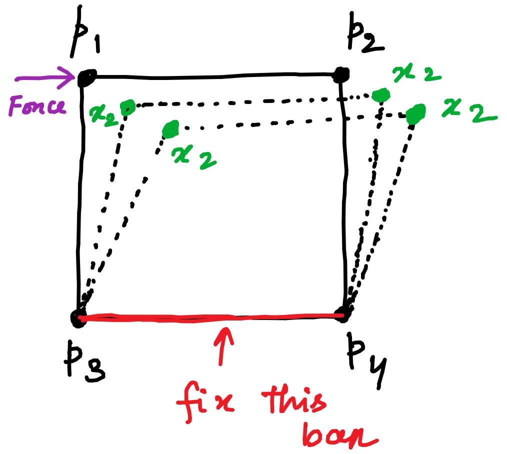
Lecture - 23

Rigid Formation

Formation $\longrightarrow$ Bar - Joint structure

(communication links $\updownarrow$ Bar; agents $\updownarrow$ Joint)

Bar-Joint structure $\downarrow$ Rigid, $\downarrow$ Flexible



(Flexible structure)

For a given bar length set

$$D = \{ \underbrace{d_{12}, d_{13}, d_{34}, d_{24}}_{\|p_i - p_j\| = d_{ij}} \}$$

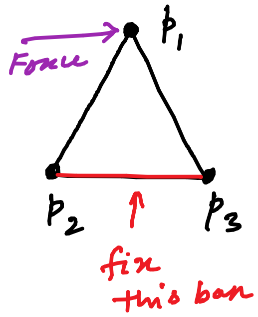
p_i : original position of the joints.

After applying force, the new positions of the joints are x_i.

There are many x_i's which will satisfy the bar length constraints :

$$\begin{cases} \|x_1 - x_2\| = d_{12} \\ \|x_1 - p_3\| = d_{13} \\ \|x_2 - p_4\| = d_{24} \\ \|p_3 - p_4\| = d_{24} \end{cases}$$

All the set of points x_1 & x_2 correspond to a set of rhombi.



(Rigid Structure)

Now for this structure, the joint at p_1 can not take a new position.

This structure also satisfied bar length constraints $\|x_i - x_j\| = d_{ij}$.

- The triangular shape can not be changed.

- If previous case we could change the shape of square to rhombus.

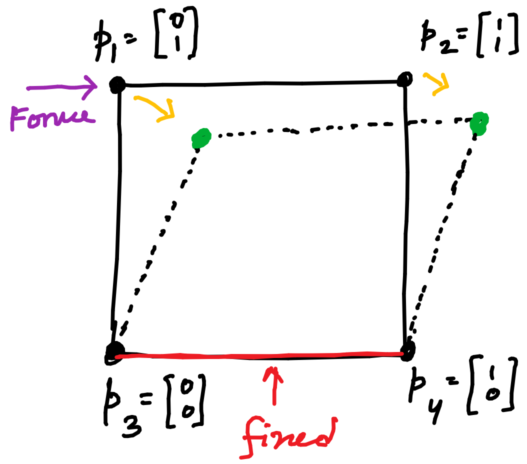
$\Downarrow$

- Above two structures are structurally different, even though both satisfy the bar-length constraints.

Bar-joint Structure
- Flexible (change the original shape)
- Rigid (Do not change the original shape)

Rigid: Needs to be defined appropriately.

→ Consider the square structure



- Fix the two points p_3 & p_4 & allow the other two points (p_1 & p_2) to take any positions (x_1, x_2), satisfying the bar-length constraint imposed on them.

We have following set of bar-length constraints.

$$\left.\begin{aligned} \|x_1 - x_2\| &= 1 \\ \|x_1 - p_3\| = \|x_1\| &= 1 \\ \|x_2 - p_4\| &= 1 \end{aligned}\right\}$$

Positions of x_1 & x_2 are determined by solving these set of equations.

From $\|x_1 - x_2\|_2^2 = 1$

$\Rightarrow (x_1 - x_2)^T (x_1 - x_2) = 1$

$\Rightarrow \underbrace{x_1^T x_1}_{=\|x_1\|=1} - 2x_1^T x_2 + x_2^T x_2 = 1$

$\Rightarrow x_2^T x_2 = 2x_1^T x_2 \quad \cdots\cdots (1)$

$\|x_2 - p_4\|_2^2 = 1$

$\Rightarrow x_2^T x_2 = 2x_2^T p_4 \quad \cdots\cdots (2)$

Let $x_1 = \begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix}$ $\quad x_2 = \begin{bmatrix} x_{2_1} \\ x_{2_2} \end{bmatrix}$

$\|x_1\|^2 = 1$

$\Rightarrow x_{1_1}^2 + x_{1_2}^2 = 1$

Let t be a parameter & set $x_{1_1} = t$

Then $x_{1_2} = \sqrt{1 - t^2}$

$t \in [0, 1]$

Then From ① & ②

$p_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$x_2^T (x_1 - p_4) = 0$$

$$\Rightarrow [x_{2_1} \;\; x_{2_2}] \begin{bmatrix} t-1 \\ \sqrt{1-t^2} \end{bmatrix} = 0$$

$$\Rightarrow x_{2_1}(t-1) + x_{2_2}(\sqrt{1-t^2}) = 0$$

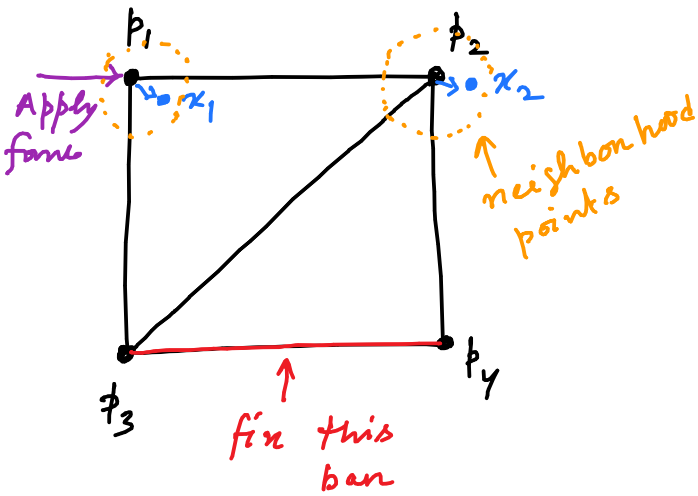
Let $x_{2_1} = 1+t \Rightarrow x_{2_2} = \sqrt{1-t^2}$

$$\left.\right\} \quad x_1 = \begin{bmatrix} t \\ \sqrt{1-t^2} \end{bmatrix} \qquad x_2 = \begin{bmatrix} 1+t \\ \sqrt{1-t^2} \end{bmatrix} \qquad t \in [0, 1]$$

For $t=0$, $x_1 = p_1$, $x_2 = p_2$.

→ The set of rhombi (due to x_1 & x_2) is characterized by the solution set of bar-length equations.

→ <u>Another Structure</u>



Bar-length equations

$$\|x_1 - x_2\|_2^2 = 1$$

$$\|x_1 - p_3\|^2 = \|x_1\|^2 = 1$$

$$\|x_2 - p_4\|^2 = 1$$

$$\|x_2 - p_3\|^2 = \|x_2\|^2 = 2$$

↑ Extra constraint.

$p_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$p_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$p_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$p_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x_1^T x_1 = 1$ & $x_2^T x_2 = 2$

$x_{1_1}^2 + x_{1_2}^2 = 1$ $\Rightarrow x_{2_1}^2 + x_{2_2}^2 = 2$

From ③

$1 + x_{2_2}^2 = 2 \Rightarrow x_{2_2} = \pm 1$

$$(x_1^T - x_2)^T (x_1 - x_2) = 1$$

$$\Rightarrow x_2^T x_1 = 1 \quad \cdots\cdots ①$$

$$(x_2 - p_4)^T (x_2 - p_4) = 1$$

$$\Rightarrow x_2^T p_4 = 1 \quad \cdots\cdots ②$$

$$\Rightarrow [x_{2_1} \ \ x_{2_2}] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$$

$$\Rightarrow x_{2_1} = 1 \quad \cdots\cdots ③$$

$$x_{2_1} x_{1_1} + x_{2_2} x_{1_2} = 1$$

$\Rightarrow x_{1_1} + x_{1_2} = 1 \leftarrow x_{2_2} = 1$

& $x_{1_1} - x_{1_2} = 1 \leftarrow x_{2_2} = -1$.

For $\left.\begin{aligned} x_{1_1} + x_{1_2} &= 1 \\ x_{1_1}^2 + x_{1_2}^2 &= 1 \end{aligned}\right\}$

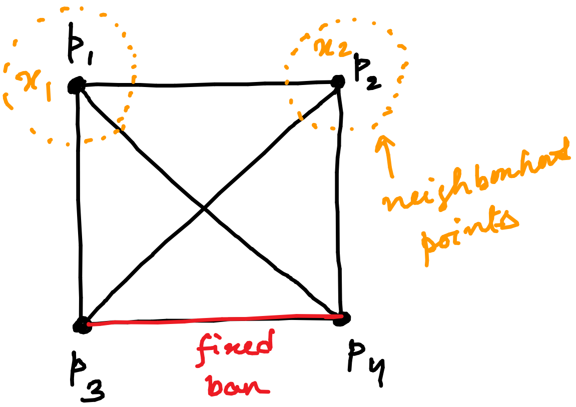
$x_{1_1} = 1, \ x_{1_2} = 0$

$x_{1_1} = 0, \ x_{1_2} = 1$

- Considering only the neighborhood points of p_1 & p_2, we have $x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\qquad\qquad p_1 \qquad\qquad p_2$

→ Consider another structure, where all 4 joints are connected by bar.



Bar-length constraints

$$\|x_1 - x_2\|^2 = 1$$

$$\|x_1 - p_3\|^2 = \|x_1\|^2 = 1$$

$$\|x_2 - p_4\|^2 = 1$$

$$\|x_2 - p_3\|^2 = \|x_2\|^2 = 2$$

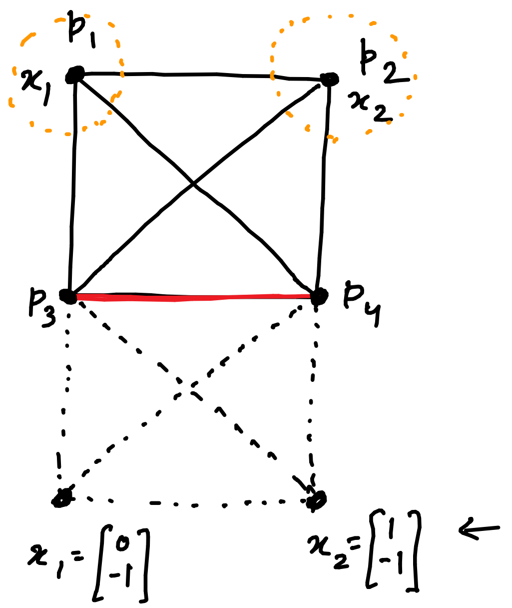
$$\|x_1 - p_4\|^2 = 2$$

The solution

$$x_{1_1} = 0 \qquad x_{2_1} = 1$$
$$x_{1_2} = \pm 1 \qquad x_{2_2} = \pm 1$$

$$\xrightarrow[\text{neighborhood points of } p_1 \,\&\, p_2]{\text{Considering}} \quad x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$p_1 \qquad\qquad p_2$$



Not neighborhood pt.

Compare

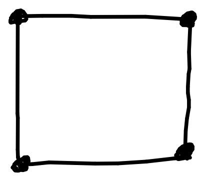


Fig-1

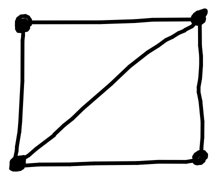


Fig-2

with

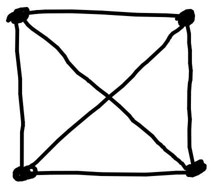


Fig-3

→ The solution in the neighborhood of pt. p_1 & p_2 for Fig-1 is in the parameterized form (parameterizes all rhombi), whereas the solution of Fig-2 & Fig-3 are same.

→ Fig-3 is always a rigid structure. Since the sol^n of Fig-2 also matches with Fig-3, Fig-2 structure is also rigid.

→ Representation of Bar-Joint structure with graphs G.

Bar $\longleftrightarrow$ Edge of a graph

Joint $\leftrightarrow$ Vertices of a graph

We also need to provide some co-ordinates to the vertices of the graphs

$\downarrow$

Embed the graph in $\mathbb{R}^n$-space, & assign a co-ordinate to the vertices of the graph

$$v_i \leftarrow p_i \in \mathbb{R}^n$$

Each vertex v_i is a point p_i in $\mathbb{R}^n$.

$\downarrow$

We name it as "Framework".

$\downarrow$ denote

$$(G, p)$$

(vertices, Edges) location of vertices in $\mathbb{R}^n$

→ Consider a framework (G, p) with edge-length equation

$$\| p_i - p_j \| = d_{ij}$$

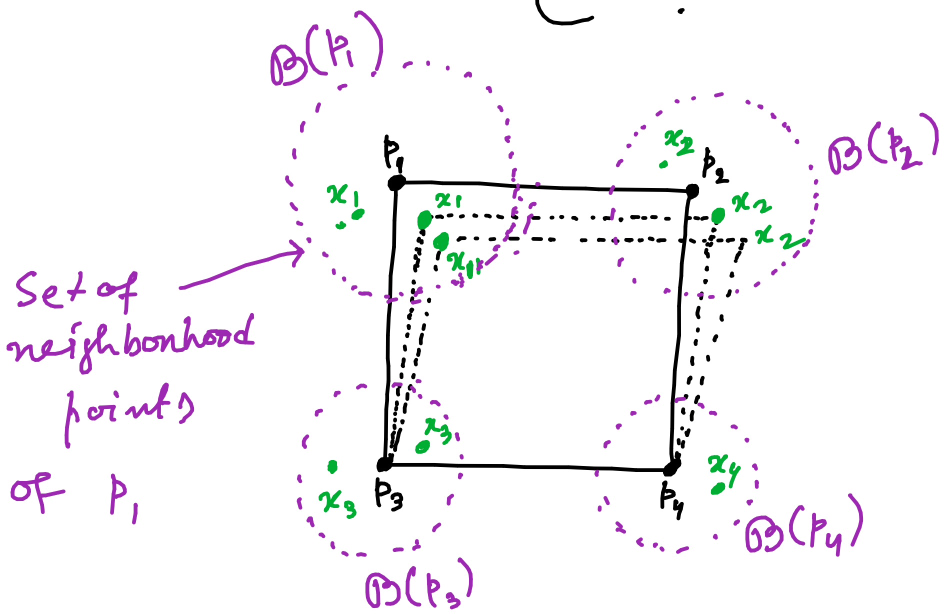
$$\updownarrow \quad \updownarrow$$

$$v_i \quad v_j$$

where, the vertices v_i & v_j are adjacent in G.

Let the set of neighborhood points of p_i be x_i, which satisfy the edge-length equation, i.e.

$$\begin{cases} \vdots \\ \| x_i - x_j \| = d_{ij} \\ \vdots \end{cases}$$



The neighborhoods points of p_i is $\mathcal{B}(p_i)$.

All points inside $\mathcal{B}(p_i)$ must satisfy the edge-length constraint.

Let the framework (G, p) has n-vertices. $p_i \in \mathbb{R}^n$

Define the following set :

$$S_G := \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{nn} \; : \; \|x_i - x_j\| = d_{ij} \text{ for all adjacent vertices } v_i \,\&\, v_j \right\}$$

May contain points which are not in the neighborhood of p_i.

Satisfy edge-length equation.

For a given framework, the neighborhood points of $p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$ is $\mathcal{B}(p)$. $p_i \in \mathbb{R}^n$

Then

$\mathcal{B}(p) \cap S_G = \Bigg\{$
(i) the points obtained by translation & rotation of the framework.
+
(ii) the points obtained due to the different shapes of the framework, which preserves only edge length

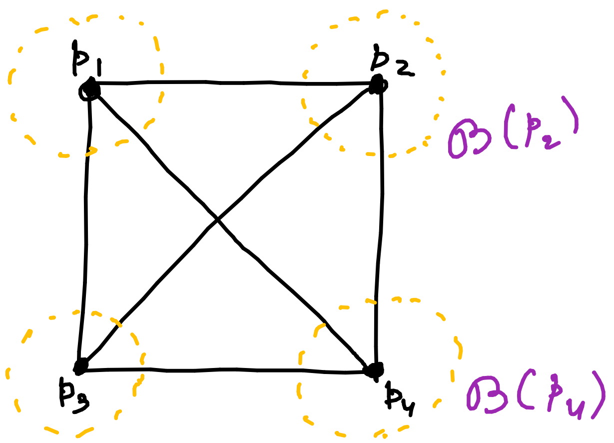
Now for a given framework (G, p), consider the corresponding complete graph G_K

In G_K, every pair of vertices are connected by edges.

Here, we have more number of edge-length equations, in comparison to G.

Let

$$S_{G_K} = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \;\middle|\; \|x_i - x_j\| = d_{ij} \text{ for all vertices } v_i \,\&\, v_j \text{ in } G_K \right\}$$



For this case, The neighborhood points are obtained only due to the translation & rotation of the framework. The shape will not change.

$$\mathcal{B}(p) \cap S_{G_K} = \left\{ \begin{array}{l} \text{the set of points that are} \\ \text{obtained only due to} \\ \text{translation \& rotation of the} \\ \text{framework.} \end{array} \right.$$

(No shape change points)

→ In general

$$\left(\mathcal{B}(p) \cap S_G\right) \supseteq \left(\mathcal{B}(p) \cap S_{G_K}\right)$$

- We will say that a framework is "Rigid" if

$$\left(\mathcal{B}(p) \cap S_G\right) = \left(\mathcal{B}(p) \cap S_{G_K}\right)$$

- A framework is "Flexible" if it is not Rigid.

→ Ref :

(i) "Stabilization of infinitesimally rigid formation of multi-robot n/w" by L. Krick, M.E. Broucke & B.A. Francis (IJC - 2009)

(ii) "Rigid & Flexible Framework" by B. Roth (The American Mathematical Monthly - 1981)