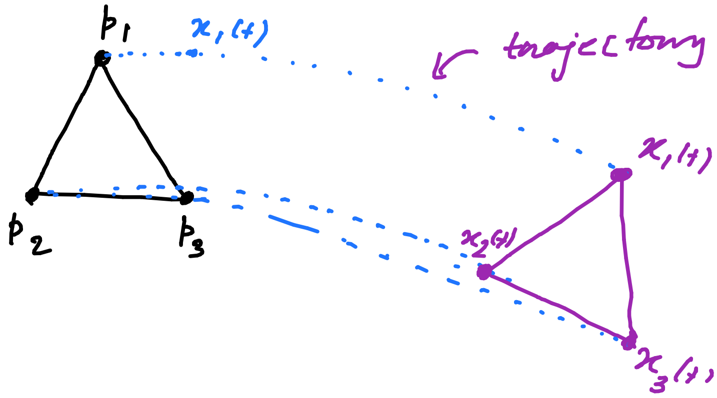
ELL 805

Lecture-24

- **Trajectory of a Framework:**

  The set of continuous points $x_1(t), x_2(t) \cdots x_n(t)$, satisfying the initial condition $x_i(0) = p_i$ & $t \geq 0$, is the trajectory of a framework.

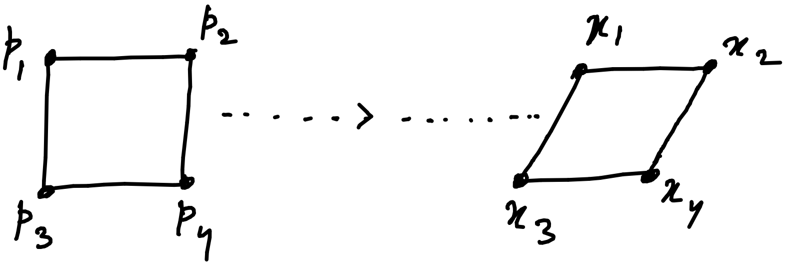


- The trajectory of a framework, which preserves the edge-lengths, i.e. $\|x_i - x_j\| = d_{ij}$, where v_i & v_j are the adjacent vertices, is called "Edge-consistent" motion.

  $\longrightarrow$ leads to Edge-consistent trajectory.

- Rigid Trajectory: The motion of a framework where the lengths between every pair of vertices remain same, is called rigid motion.

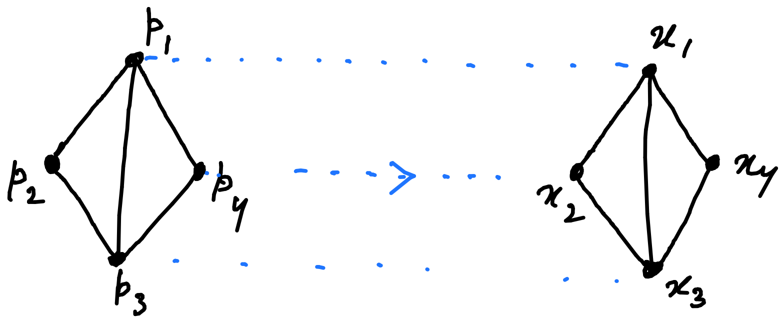
  A rigid motion leads to a Rigid trajectory.



Not rigid motion and hence, not rigid trajectory.

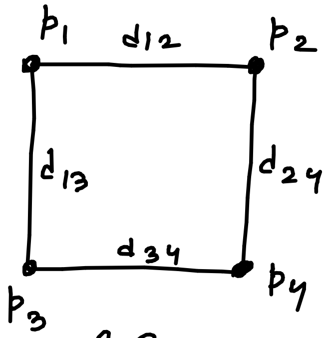
$\|x_1 - x_4\|$ & $\|x_3 - x_2\|$ have changed.

The lengths between every pair of vertices x_i & x_j do not remain same over the motion of the framework



Rigid motion, Since the lengths between every pair of vertices remain same during the motion.

→ Rigid Framework



(G, p)

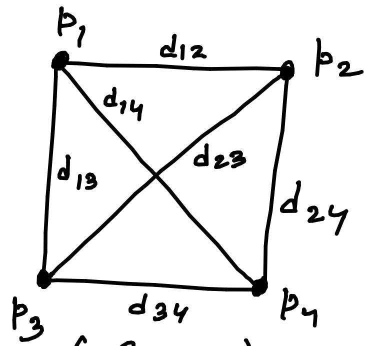
$\downarrow$

$$\begin{cases} \|x_2 - x_1\| = d_{12} \\ \|x_4 - x_2\| = d_{24} \\ \|x_4 - x_3\| = d_{34} \\ \|x_3 - x_1\| = d_{13} \end{cases}$$

$\downarrow$

$$S_G = \{ x : \|x_i - x_j\| = d_{ij} \}$$

The sol^n may contain all points which are not in the neighborhood of p_i.



(G_K, p)

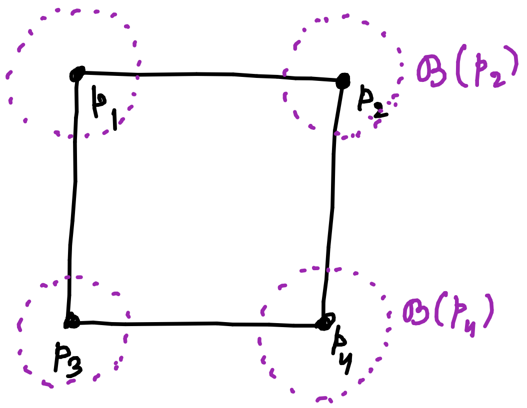
$\downarrow$

$$\left.\begin{aligned} \|x_2 - x_1\| &= d_{12} \\ \|x_3 - x_1\| &= d_{13} \\ \|x_4 - x_1\| &= d_{14} \\ \|x_3 - x_2\| &= d_{23} \\ \|x_4 - x_2\| &= d_{24} \\ \|x_3 - x_4\| &= d_{34} \end{aligned}\right\}$$

Edge-length equations

$\downarrow$

$$S_{G_K} = \{ x : \|x_i - x_j\| = d_{ij} \}$$



$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

$\mathcal{B}(p)$

In general

$$\left(\mathcal{B}(p) \cap S_G\right) \supseteq \left(\mathcal{B}(p) \cap S_{G_K}\right)$$

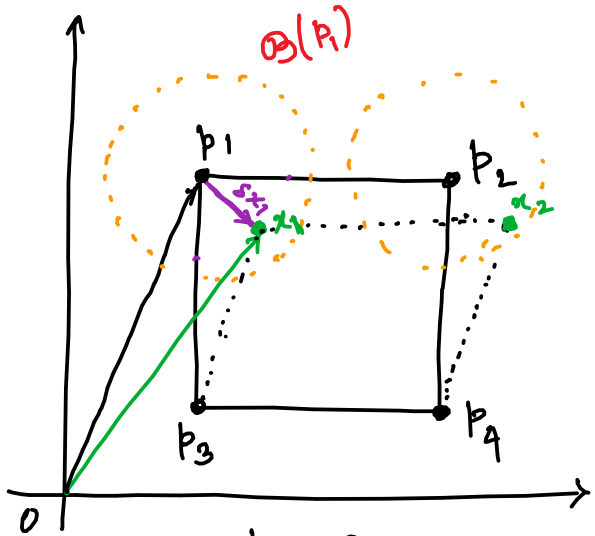
If $\boxed{\left(\mathcal{B}(p) \cap S_G\right) = \left(\mathcal{B}(p) \cap S_{G_K}\right)}$

Rigid Framework

→ Infinitesimal Rigidity

Let say we are interested in characterizing all neighborhood points x_i of $p_i \in \mathbb{R}^n$, which preserves the specified edge-lengths.

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \qquad p_i \in \mathbb{R}^n$$



$$p_1 + \delta x_1 = x_1$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad x_i \in \mathbb{R}^n$$

↑ neighborhood point of p

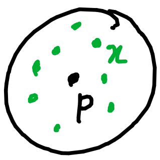
$$\boxed{p + \delta x = x}$$

Define a function

$$g_G(x) = \begin{bmatrix} \vdots \\ \|x_i - x_j\|^2 \\ \vdots \end{bmatrix}$$ for a given framework (G, p)

Rigidity function

Let x be a point in the neighborhood of point p.



Then, each point $x = p + \delta x$

where δx is a vector. ($x_1 = p_1 + \delta x_1$ as in fig)

Linearize the Rigidity fnt^n $g_G(x)$ about the point p:

$$\text{(*)} \cdots \boxed{g_G(x) = g_G(p + \delta x) = g_G(p) + J_G(p)\,\delta x}$$

Neglecting the higher order terms.

$g_G(x)$ is a multivariate fnt^n, & hence

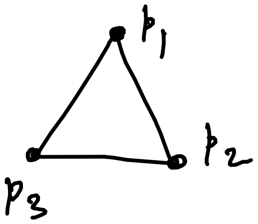
$J_G(p)$ is a matrix, which is the Jacobian of $g_G(x)$.

$$J_G(p) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_{nR}} \\ \vdots & & \ddots & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_{nR}} \end{bmatrix}\Bigg|_p$$ (evaluated at point p)

The equation (*) is an expression for all neighborhood points x of p. We need only those neighborhood points x, which satisfy the edge-length equation,

i.e. $$\boxed{g_G(x) = g_G(p+\delta x) = g(p)} \quad \cdots \; (**)$$

For instance



$$g(p+\delta x) = g(x) = \begin{bmatrix} \|x_1 - x_2\|^2 \\ \|x_2 - x_3\|^2 \\ \|x_3 - x_1\|^2 \end{bmatrix} = g(p) = \begin{bmatrix} \|p_1 - p_2\|^2 \\ \|p_2 - p_3\|^2 \\ \|p_3 - p_1\|^2 \end{bmatrix}$$

→ The relation (**) can be achieved in (*) by ensuring that

$$\boxed{J_G(p)\,\delta x = 0}$$

$J_G(p)$: Rigidity Matrix

Note that $J_G(p)$ is a real matrix, since it is evaluated at p.

$$\Downarrow$$

To ensure $J_G(p)\,\delta x = 0$, the vector

$$\delta x \in \mathcal{N}(J_G(p)).$$

↑ null space

Any vector δx that belongs to the null space of Rigidity matrix i.e. $\delta x \in \mathcal{N}(J_G(p))$

$$\Downarrow$$

$$J_G(p)\,\delta x = 0$$

For a given framework (G, p), one can construct the corresponding complete graph framework (G_K, p)

- For (G_K, p), to preserve the edge-length constraints, the neighborhood points x of p to be chosen s.t. the vector $\boxed{\delta x \in \mathcal{N}(J_{G_K}(p))}$

$J_{G_K}(p)$: Rigidity matrix corresponding to the complete graph.

Since the number of edge-length equations for (G_K, p) is greater than (G, p), we have

$$\boxed{\mathcal{N}(J_G(p)) \supseteq \mathcal{N}(J_{G_K}(p))}$$

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We noticed that

$$g_G(x) = g_G(p+\delta x) = g_G(p) \quad \text{iff}$$

$$J_G(p)\,\delta x = 0$$

i.e. $\quad \delta x \in \mathcal{N}(J_G(p))$

- Instead of looking at the perturbation vector, we look at the rate of change of perturbations of each vertices of the framework

$\downarrow$

$$\lim_{\delta t \to 0} \frac{\delta x_i}{\delta t} \leftarrow \text{denote it as } u_i$$

So $\quad u_i = \lim_{\delta t \to 0} \dfrac{\delta x_i}{\delta t}$

Hence we can assign a velocity vector u_i to each vertices of the framework, & look at the velocity vector $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, which satisfy

$$J_G(p)\,u = 0$$

i.e. $\quad \boxed{u \in \mathcal{N}(J_G(p))}$

We had discussed that

$$\mathcal{N}\left(J_G(p)\right) \supseteq \mathcal{N}\left(J_{G_K}(p)\right)$$

complete graph G_K

- We say that the framework (G, p) is "infinitesimally rigid" if

$$\boxed{\mathcal{N}\left(J_G(p)\right) = \mathcal{N}\left(J_{G_K}(p)\right)}$$

→ The null-space of Rigidity matrix $\mathcal{N}(J_G(p))$ plays an important role in determining the infinitesimal rigidity of a framework.

→ Computation of $J_G(p)$

Let $x_i = \begin{bmatrix} x_{i_1} \\ x_{i_2} \end{bmatrix}$ $\qquad x_i \in \mathbb{R}^2$

$x_j = \begin{bmatrix} x_{j_1} \\ x_{j_2} \end{bmatrix}$

The rigidity funct^n $g_G(x) = \begin{bmatrix} \vdots \\ g_i(x) = \|x_i - x_j\|_2^2 \\ \vdots \end{bmatrix}$

$$g_i(x) = \|x_i - x_j\|_2^2 = (x_i - x_j)^T (x_i - x_j)$$

$$= (x_{i_1} - x_{j_1})^2 + (x_{i_2} - x_{j_2})^2$$

g_i is a func^n of $x_{i_1}, x_{i_2}, x_{j_1}, x_{j_2}$

So

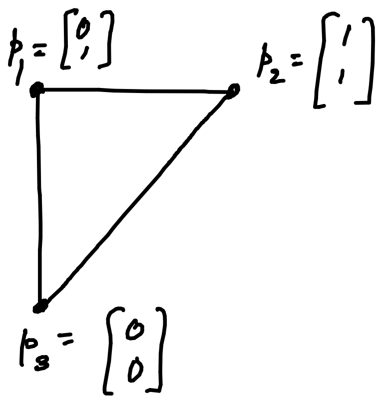
$$\frac{\partial g_i}{\partial x} = \begin{bmatrix} \frac{\partial g_i}{\partial x_{i_1}} & \frac{\partial g_i}{\partial x_{i_2}} & \frac{\partial g_i}{\partial x_{j_1}} & \frac{\partial g_i}{\partial x_{j_2}} \end{bmatrix}$$

$$= 2\begin{bmatrix} \underbrace{x_{i_1} - x_{j_1} \quad x_{i_2} - x_{j_2}} & \underbrace{-(x_{i_1} - x_{j_1}) \quad -(x_{i_2} - x_{j_2})} \end{bmatrix}$$

$$= 2\begin{bmatrix} \underbrace{(x_i - x_j)^T}_{\frac{\partial g_i}{\partial x_i}} & \underbrace{(x_j - x_i)^T}_{\frac{\partial g_i}{\partial x_j}} \end{bmatrix}$$

Hence

$$\left.\frac{\partial g_i}{\partial x}\right|_p = 2\begin{bmatrix} (p_i - p_j)^T & (p_j - p_i)^T \end{bmatrix}$$



$$g_G(x) = \begin{array}{c} \begin{bmatrix} \|x_1 - x_2\|_2^2 \\ \|x_2 - x_3\|_2^2 \\ \|x_3 - x_1\|_2^2 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow g_1 \\ \leftarrow g_2 \\ \leftarrow g_3 \end{array}$$

Then

$$J_G(p) = \left.\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \\ \frac{\partial g_3}{\partial x_1} & \frac{\partial g_3}{\partial x_2} & \frac{\partial g_3}{\partial x_3} \end{bmatrix}\right|_{(p)}$$

$$= 2 \left.\begin{bmatrix} (x_1 - x_2)^T & (x_2 - x_1)^T & 0 \\ 0 & (x_2 - x_3)^T & (x_3 - x_2)^T \\ (x_1 - x_3)^T & 0 & (x_3 - x_1)^T \end{bmatrix}\right|_{(p)}$$

$$= 2 \underbrace{\begin{bmatrix} (p_1 - p_2)^T & (p_2 - p_1)^T & 0 \\ 0 & (p_2 - p_3)^T & (p_3 - p_2)^T \\ (p_1 - p_3)^T & 0 & (p_3 - p_1)^T \end{bmatrix}}_{\text{Rigidity matrix}}$$

$$= 2 \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

- Assigning the velocity vectors to the vertices.

We need $\boxed{J_G(p)\, u = 0}$

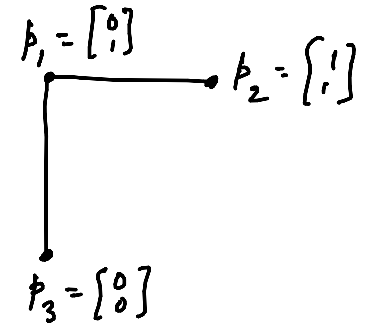
Here
$$\begin{bmatrix} (p_1-p_2)^T & (p_2-p_1)^T & 0 \\ 0 & (p_2-p_3)^T & (p_3-p_2)^T \\ (p_1-p_3)^T & 0 & (p_3-p_1)^T \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0$$

We have to look at the $\mathcal{N}(J_G(p))$

$$\text{rank}\underbrace{\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}}_{J_G(p)} = 3$$

Then $\dim\big(\mathcal{N}(J_G(p))\big) = 3$

- For the framework



$$J_G(p) = \begin{bmatrix} (p_1-p_2)^T & (p_2-p_1)^T & 0 \\ (p_1-p_3)^T & 0 & (p_3-p_1)^T \end{bmatrix}$$

(Obtained by removing the row $\frac{\partial g_2}{\partial x}$)

- $J_G(p)$ can be obtained by removing the rows associated with deleted edges of complete graph G_K.

$$\text{rank}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} = 2$$

$\Rightarrow \dim \mathcal{N}(J_G(p)) = 4$

For a given framework (G, p) with its corresponding framework (G_K, p) where G_K is the complete graph associated with G.

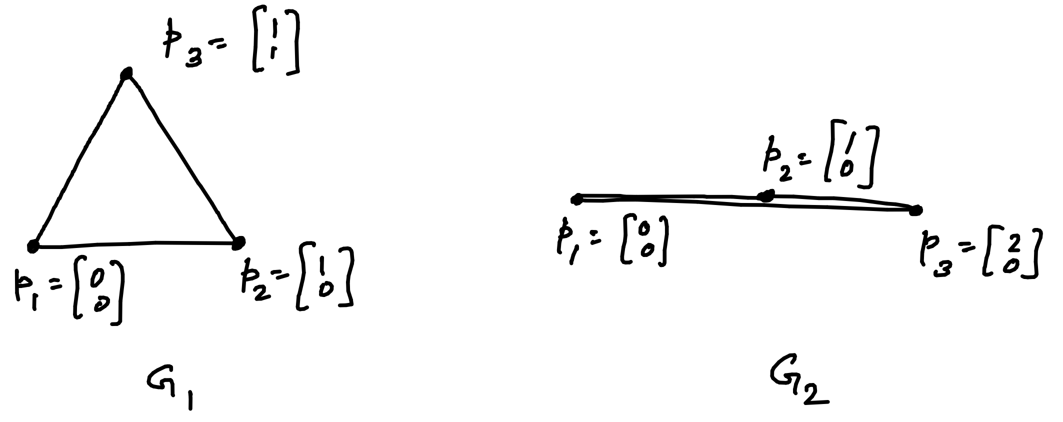
$\downarrow$

- The number of rows in $J_G(p)$ is less than or equal to $J_{G_K}(p)$, whereas both have same number of columns. Hence, in general

$$\mathcal{N}(J_{G_K}(p)) \subseteq \mathcal{N}(J_G(p))$$

→ <u>Some Results</u>

- If a framework is infinitesimally rigid, then it is also rigid. (The reverse may not always be true)



Both frameworks (G_1, p) & (G_2, p) are rigid, since they are embedding of complete graphs in $\mathbb{R}^2$.

The rigidity matrices :

$$J_{G_1}(p) = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 \end{bmatrix} .$$

$$\text{rank}\,(J_{G_1}(p)) = 3$$

$$J_{G_2}(p) = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

$$\text{rank}\,(J_{G_2}(p)) = 2$$

Hence $(G_1, p) \rightarrow$ Infinitesimally rigid

$(G_2, p) \rightarrow$ Not infinitesimally rigid.

- <u>Rigidity Matrix Result</u>

Assume that the MAS has n-agents & the initial vertices are at $p_i \in \mathbb{R}^2$.

$\Downarrow$

Then, a framework (G, p) is infinitesimally rigid if and only if

$$\boxed{\text{rank}\,(J_G(p)) = 2n-3}$$

- A framework (G, p) is "**minimally rigid**" if it is rigid but does not remain rigid after the removal of any single edge from G.

Rigid
Minimally rigid

Rigid
Not minimally rigid.

- A framework (G, p) with n-vertices in $\mathbb{R}^2$ is minimally rigid if and only if the following conditions hold:

  (i) G has $2n-3$ edges, and

  (ii) each induced subgraph, induced by the vertices with index $n' \leq n$, has no more than $2n'-3$ edges.