Lecture - 26

Let the MAS has n- robots/agents.

The target formation is specified as

(G, d) $G \rightarrow$ Network graph

$d = \begin{bmatrix} \vdots \\ d_{ij} \\ \vdots \end{bmatrix}$ where v_i & v_j are adjacent in G

d_{ij} : distance between i^{th} & j^{th} agent.

- The robots are on the plane & they are capable of measuring the relative positions w.r.t. their neighbours.

If robot-1 can see Robot 2 & robot 5 then following measurements are available to robot-1 : $x_2 - x_1$ & $x_5 - x_1$.

$\rightarrow$ To achieve the target formation (infinitesimally rigid) we propose gradient based feedback control.

The dynamics of each robot is

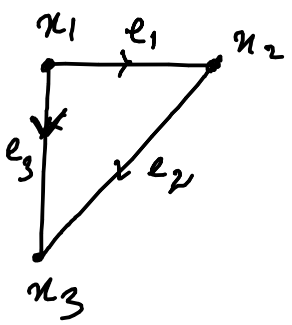
$$\dot{x}_i = u_i \qquad x_i \in \mathbb{R}^2$$
$$u_i \in \mathbb{R}^2$$

d_{ij} is specified distance bet^n ith & jth robot.

Define some error vector

$$e_i = x_j - x_i$$

We need $x_j - x_i \longrightarrow d_{ij}$



$$\left.\begin{aligned} e_1 &= x_2 - x_1 \\ e_2 &= x_3 - x_2 \\ e_3 &= x_3 - x_1 \end{aligned}\right\} \qquad e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = D^T x$$

when D is incidence matrix of G.

$$D = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$e = D^T x = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\leftarrow$ written for $x_i \in \mathbb{R}$

For $x_i \in \mathbb{R}^2 \rightarrow e_1 = \begin{bmatrix} e_{1_1} \\ e_{1_2} \end{bmatrix} = x_2 - x_1 = \begin{bmatrix} x_{2_1} \\ x_{2_2} \end{bmatrix} - \begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix}$

$$e = \begin{bmatrix} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \\ e_{31} \\ e_{32} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \\ x_2 \\ \\ x_3 \\ \\ \end{bmatrix}$$

$$= \begin{bmatrix} -1 \cdot I & 1 \cdot I & 0 \cdot I \\ 0 \cdot I & -1 \cdot I & 1 \cdot I \\ -1 \cdot I & 0 \cdot I & 1 \cdot I \end{bmatrix} \begin{bmatrix} x \end{bmatrix}$$

$$= \underbrace{(D^T \otimes I)}_{\hat{D}^T} x$$

$\otimes$: Kronecker Product

→ Rendezvous Problem

$d_{ij} = 0$ for all i, j

$$d = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

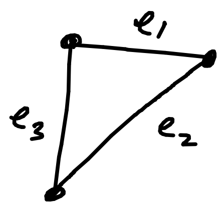
The control law $u = -Lx$ L : Laplacian Matrix

$x \in \mathbb{R}$ $\dot{x} = -Lx$

For $x \in \mathbb{R}^2$ $\dot{x} = -(L \otimes I)x$

With this control law, we will achieve consensus, that is all robots reach to a common point.

$$\boxed{u = -Lx}$$



$$\varphi(x) = \tfrac{1}{2}\|e\|_2^2 = e_1^2 + e_2^2 + e_3^2$$

$$= \tfrac{1}{2}\left[(x_2 - x_1)^2 + (x_3 - x_2)^2 + (x_3 - x_1)^2\right]$$

$$\left.\begin{aligned}
\frac{\partial \varphi}{\partial x_1} &= (x_2 - x_1)(-1) + (x_3 - x_1)(-1) \\
&= x_1 - x_2 + x_1 - x_3 \\
&= 2x_1 - x_2 - x_3 \\
\frac{\partial \varphi}{\partial x_2} &= x_2 - x_1 + x_2 - x_3 \\
&= 2x_2 - x_1 - x_3 \\
\frac{\partial \varphi}{\partial x_3} &= x_3 - x_2 + x_3 - x_1 \\
&= 2x_3 - x_1 - x_2
\end{aligned}\right\}
\underbrace{\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}}_{L}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x}$$

Gradient of $\varphi(x) = \nabla \varphi(x) = \left[\frac{\partial \varphi(x)}{\partial x}\right]^T$

$$= \left[\frac{\partial \varphi}{\partial x_1} \quad \frac{\partial \varphi}{\partial x_2} \quad \frac{\partial \varphi}{\partial x_3}\right]^T$$

$$\Rightarrow \boxed{\nabla \varphi(x) = Lx}$$

$$\varphi(x) = \frac{1}{2}\|e\|_2^2 = \frac{1}{2}\|D^T x\|_2^2 \qquad (e = D^T x)$$

$$= \frac{1}{2} x^T D D^T x$$

$$= \frac{1}{2} x^T L x \qquad L = DD^T$$

Result : for a matrix A, & vector x

$$\varphi(x) = x^T A x \rightarrow \frac{\partial \varphi}{\partial x} = x^T (A^T + A)$$

$$\nabla \varphi(x) = \left[\frac{\partial \varphi}{\partial x}\right]^T = (A^T + A) x$$

For our case $\varphi(x) = \frac{1}{2} x^T L x$ & L is symmetric

$$\boxed{\nabla \varphi(x) = Lx}$$

The feedback control we have suggested

$$\boxed{u = -Lx = -\nabla \varphi(x)}$$

when $\varphi(x) = \frac{1}{2}\|x_i - x_j\|_2^2$

→ For $d \neq 0$

We define a funct'

$$\bar{d} = \begin{bmatrix} \vdots \\ d_{ij}^2 \\ \vdots \end{bmatrix} \qquad \boxed{\varphi(x) = \frac{1}{2}\|g_G(x) - \bar{d}\|_2^2} \qquad g_G = \begin{bmatrix} \vdots \\ \|x_i - x_j\|_2^2 \\ \vdots \end{bmatrix}$$

Rigidity funct'.

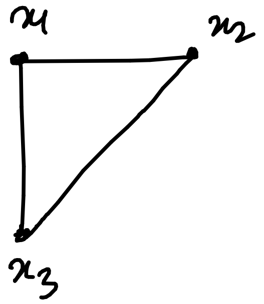
The fnt $\varphi(x) = \frac{1}{2} \| g_G(x) - \bar{d} \|_2^2$ is a positive semi-definite fnt?

Can be considered as Lyapunov function candidate for formation stability.

& $\varphi(x) = 0$ when $g_G(x) = d$

$$\Downarrow$$

$$\| x_i - x_j \|_2 = d_{ij}$$



$$g_G(x) = \begin{bmatrix} \| x_1 - x_2 \|_2^2 \\ \| x_2 - x_3 \|_2^2 \\ \| x_3 - x_1 \|^2 \end{bmatrix}$$

$$g_G(x) - d = \begin{cases} \| x_1 - x_2 \|_2^2 - d_{12}^2 \\ \| x_2 - x_3 \|_2^2 - d_{23}^2 \\ \| x_3 - x_1 \|_2^2 - d_{31}^2 \end{cases}$$

Control law $\boxed{u = - \nabla \varphi(x)}$

$$\boxed{\dot{x}_i = u_i = - \sum_{j \in N_i} \frac{1}{2} (\| x_i - x_j \|^2 - d_{ij}^2) (x_i - x_j)}$$

N_i : Neighborhood set

$$\varphi(x) = \frac{1}{2}\|g_G(x) - \bar{d}\|_2^2 = \frac{1}{2}\underbrace{(g_G(x) - \bar{d})^T}_{h^T(x)}\underbrace{(g_G(x) - \bar{d})}_{h(x)}$$

So $\varphi(x)$ can be written as

$$\varphi(x) = \frac{1}{2} h^T h$$ where h is a funct^n of vector x

$$\frac{\partial \varphi}{\partial x} = \frac{1}{2} \cdot 2 h^T \frac{\partial h}{\partial x}$$

$$\Rightarrow \frac{\partial \varphi}{\partial x} = (g_G(x) - \bar{d})^T \frac{\partial g_G(x)}{\partial x}$$

$$= (g_G(x) - \bar{d})^T J_G(x)$$

Jacobian of the Rigidity funct^n. (Rigidity Matrix)

$$\nabla \varphi(x) = \left[\frac{\partial \varphi}{\partial x}\right]^T$$

$$= (J_G(x))^T (g_G(x) - \bar{d})$$

→ For further study on this, please refer to

- Stabilization of infinitesimally rigid formations of multi-robot network, (IJC - 2009) by L. Krick, M.E. Broucke & B.A. Francis

- Some useful results on matrix calculus.

Let $x \in \mathbb{R}^n$ $y \in \mathbb{R}^m$ & $y = \varphi(x)$

Then,

$$\frac{\partial y}{\partial x} = \frac{\partial \varphi}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & & \ddots & \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}_{m \times n}$$

- For $y = Ax$ $A \in \mathbb{R}^{m \times n}$ is a real matrix & does not depend on x.

$$\boxed{\frac{\partial y}{\partial x} = A}$$

- Let x is $n \times 1$ & it is a fn^n of vector z & A is independent of z.

Then for $y = A\,x(z)$

$$\boxed{\frac{\partial y}{\partial z} = A \frac{\partial x}{\partial z}}$$

- Let $\varphi(x, y) = y^T A x$ $x \rightarrow n \times 1$, $y \rightarrow m \times 1$

$$\boxed{\frac{\partial \varphi}{\partial x} = y^T A \quad \& \quad \frac{\partial \varphi}{\partial y} = x^T A^T}$$

- For $\varphi(x) = x^T A x$

$$\boxed{\frac{\partial \varphi}{\partial x} = x^T (A + A^T)}$$

- For $\varphi(z) = y^T x$ where y & x are funct's of vector z

$$\boxed{\frac{\partial \varphi}{\partial z} = x^T \frac{\partial y}{\partial z} + y^T \frac{\partial x}{\partial z}}$$

- For $\varphi(z) = x^T x$ x is a funct of z

$$\boxed{\frac{\partial \varphi}{\partial z} = 2x^T \frac{\partial x}{\partial z}}$$

- For $\varphi(z) = y^T A x$ x & y are funct's of z, but A is not funct of z.

$$\boxed{\frac{\partial \varphi}{\partial x} = x^T A^T \frac{\partial y}{\partial z} + y^T A \frac{\partial x}{\partial z}}$$

- For $\varphi(z) = x^T A x$ x funct of z

$$\boxed{\frac{\partial \varphi}{\partial z} = x^T (A + A^T) \frac{\partial x}{\partial z}}$$