ELL 805

Lecture - 27

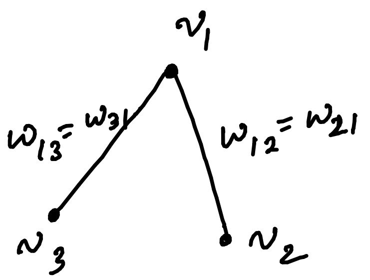
- Consensus for general LTI Agents

Agent dynamics : $\begin{cases} \dot{x}_i = Ax_i + Bu_i \\ x_i(0) : \text{initial condition for } i\text{th agent.} \end{cases}$

Identical Agents $\quad i = 1, 2, \cdots, n$

$x_i \in \mathbb{R}^n \quad u_i \in \mathbb{R}^m \quad A \in \mathbb{R}^{n \times n}, \; B \in \mathbb{R}^{n \times m}$

- Assume that the network graph is connected and undirected.



- Relative state information available at ith agent :

$$\boxed{z_i(t) = \sum_{j \in \mathcal{N}(i)} w_{ij} \left(x_j(t) - x_i(t) \right)}$$

$\mathcal{N}(i)$: Neighborhood set of ith agent.

For $x_i \in \mathbb{R}$ (scalar agent dynamics)

$$z_1 = w_{12}(x_2 - x_1) + w_{13}(x_3 - x_1)$$

$$z_2 = w_{12}(x_1 - x_2)$$

$$z_3 = w_{13}(x_1 - x_3)$$

$$\Rightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -(w_{12}+w_{13}) & w_{12} & w_{13} \\ w_{12} & -w_{12} & 0 \\ w_{13} & 0 & -w_{13} \end{bmatrix}}_{-L} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

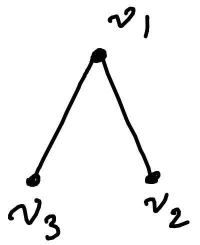
$-L$: Laplacian matrix of network graph

$$\downarrow$$

$$\boxed{z = -Lx}$$

- For $x_i \in \mathbb{R}^2 \rightarrow x_1 = \begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix} \quad x_2 = \begin{bmatrix} x_{2_1} \\ x_{2_2} \end{bmatrix}$

$$x_3 = \begin{bmatrix} x_{3_1} \\ x_{3_2} \end{bmatrix}$$



$$\underbrace{\begin{bmatrix} z_{1_1} \\ z_{1_2} \end{bmatrix}}_{z_1} = w_{12}\left(\underbrace{\begin{bmatrix} x_{2_1} \\ x_{2_2} \end{bmatrix}}_{x_2} - \underbrace{\begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix}}_{x_1} \right) + w_{13}\left(\underbrace{\begin{bmatrix} x_{3_1} \\ x_{3_2} \end{bmatrix}}_{x_3} - \underbrace{\begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix}}_{x_1} \right)$$

$$\downarrow$$

$$z_1 = \begin{bmatrix} -(w_{12}+w_{13})\cdot I_2 & w_{12}\cdot I_2 & w_{13}\cdot I_2 \end{bmatrix} \begin{bmatrix} x_{1_1} \\ x_{1_2} \\ \hdashline x_{2_1} \\ x_{2_2} \\ \hdashline x_{3_1} \\ x_{3_2} \end{bmatrix}$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly

$$z_2 = \begin{bmatrix} w_{12}\cdot I_2 & -w_{12}\cdot I_2 & 0\cdot I_2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ \hdashline x_{21} \\ x_{22} \\ \hdashline x_{31} \\ x_{32} \end{bmatrix}$$

$$z_3 = \begin{bmatrix} w_{13}\cdot I & 0\cdot I_2 & -w_{13}\cdot I_2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ \hdashline x_{21} \\ x_{22} \\ \hdashline x_{31} \\ x_{32} \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -(w_{12}+w_{13}) & w_{12}\cdot I_2 & w_{13}\cdot I_2 \\ w_{12}\cdot I_2 & -w_{12}\cdot I_2 & 0\cdot I_2 \\ w_{13}\cdot I_2 & 0\cdot I_2 & -w_{13}\cdot I_2 \end{bmatrix}}_{-L \otimes I_2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- For $x_i \in \mathbb{R}^n$, $z_i \in \mathbb{R}^n$:

$$z_i = \sum_{j \in N(i)} w_{ij}\,(x_j - x_i)$$

$$\downarrow$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{nn} \qquad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{R}^{nn}$$

Combining all

$$\boxed{z = (-L \otimes I_n)\,x}$$

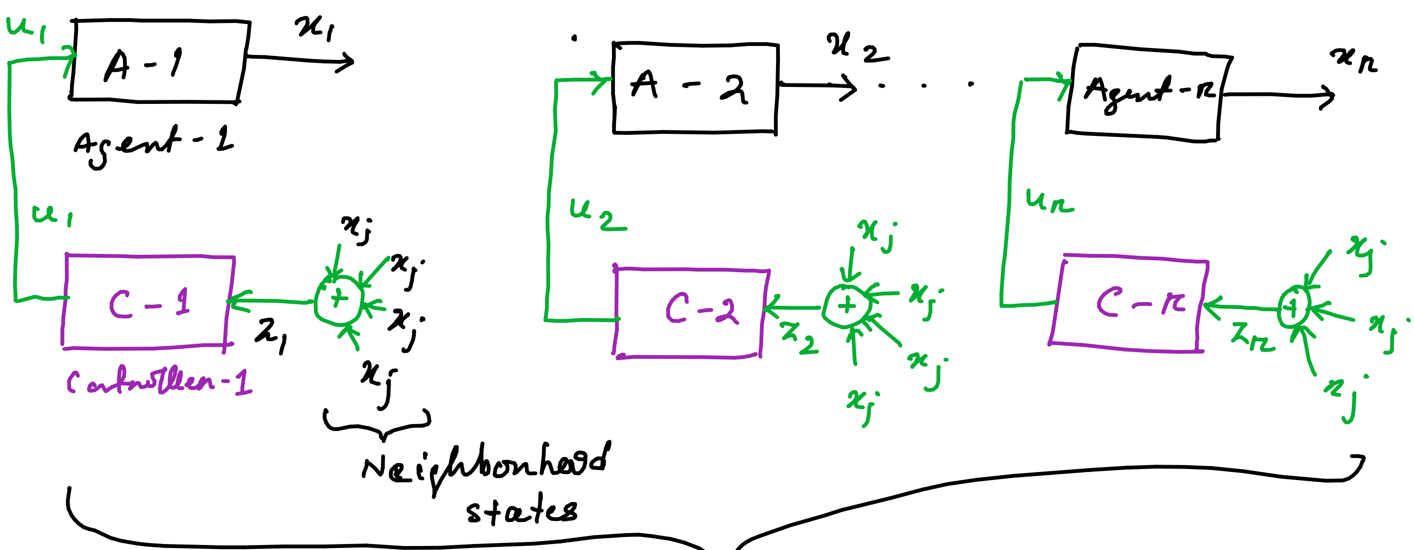
$$\underbrace{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix}A & & & 0\\ & A & & \\ & & \ddots & \\ 0 & & & A\end{bmatrix}}_{I_n \otimes A}\underbrace{\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}}_{x} + \underbrace{\begin{bmatrix}B & & \\ & B & \\ & & \ddots & \\ & & & B\end{bmatrix}}_{I_n \otimes B}\underbrace{\begin{bmatrix}u_1\\ u_2\\ \vdots\\ u_n\end{bmatrix}}_{u}$$

$$\dot{x} = (I_n \otimes A)\,x + (I_n \otimes B)\,u \quad \ldots\ldots (*)$$

Let the feedback control for ith agent be

$$\boxed{u_i = \eta F x_i} \qquad \text{for } i = 1, 2, \cdots, n$$

Design Parameters $\begin{cases} \eta : \text{is a scalar gain} \\ F : \text{Gain matrix} \end{cases}$



<u>Distributed Control Architecture</u>

Each agent has its own local controller, which takes information from neighbors, & produces control signal for the agent.

For the control laws

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \eta F & & & 0 \\ & \eta F & & \\ & & \ddots & \\ 0 & & & \eta F \end{bmatrix}}_{I_n \otimes \eta F} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$\boxed{u = (I_n \otimes \eta F)\, z}$$

Implementing u in ⊛, the closed loop system:

$$\dot{x} = (I_n \otimes A)x + (I_n \otimes B)(I_n \otimes \eta F)\, z$$

$$= (I_n \otimes A)x + (I_n \otimes \eta BF)\, z$$

$$\hookrightarrow = (-L \otimes I_n)\, x$$

$$= (I_n \otimes A)x + (I_n \otimes \eta BF)(-L \otimes I_n)\, x$$

$$= (I_n \otimes A)x + (-L \otimes \eta BF)\, x$$

$$= \underbrace{\left[(I_n \otimes A) + (-L \otimes \eta BF)\right]}_{A_c} x$$

$\boxed{\dot{x} = A_c x}$ closed loop system.

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Recall closed-loop system dynamics

$$\dot{x} = A_c x$$

where $A_c = (I_n \otimes A) + (-L \otimes \eta BF)$

Define a new matrix as follows:

$$M := \begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & & \ddots & \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & 1-\frac{1}{n} \end{bmatrix}$$

'n' is the number of agents.

can be written as:

$$M = I_n - \frac{1}{n}(\underbrace{\mathbb{1} \cdot \mathbb{1}^T}_{J})$$

Recall

$$\text{The eigenvalues of } J = \begin{cases} n, & \text{with multiplicity } 1 \\ 0, & \text{" " } n-1 \end{cases}$$

$\downarrow$

$$\text{The eigenvalues of } M = \begin{cases} 1 - n \cdot \frac{1}{n} = 0, & \text{with multiplicity } 1 \\ 1, & \text{" " } n-1 \end{cases}$$

Properties of M:

- It is symmetric
- It has only one eigenvalue at 0.
- Row & column sums to 0.

$$\downarrow$$

$$\boxed{\begin{aligned} \mathbb{1}^T M &= 0 \\ M \cdot \mathbb{1} &= 0 \end{aligned}}$$

- The eigenvector corresponding to '0' eigenvalue is $\mathbb{1}$.

- Define a new state variable $\delta(t)$ as follows:

$$\boxed{\delta(t) := (M \otimes I_n)\, x(t)} \qquad \delta \in \mathbb{R}^{nn}$$

$$\dot{\delta}(t) = (M \otimes I_n)\, \dot{x}(t)$$

$$= \underbrace{(M \otimes I_n)}\, \underbrace{A_c}\, x(t)$$

These two matrices commute

i.e. $(M \otimes I_n) A_c = A_c (M \otimes I_n)$

$$(M \otimes I_n) A_c = (M \otimes I_n)\left[(I_n \otimes A) + (-L \otimes \eta BF)\right]$$

$$= (M \otimes A) + (-ML \otimes \eta BF) \quad \cdots\cdots \text{(A)}$$

$$A_c(M \otimes I_n) = \left[(I_n \otimes A) + (-L \otimes \eta BF) \right] (M \otimes I_n)$$

$$= (M \otimes A) + (-LM \otimes \eta BF) \quad \cdots \quad \Delta\Delta$$

Comparing Δ & $\Delta\Delta$, we see that

$$(M \otimes I_n) A_c = A_c (M \otimes I_n) \quad \text{iff} \quad \underline{LM = ML}$$

$$LM = L\left(I_n - \tfrac{1}{n}\mathbb{1}\mathbb{1}^T\right) = L - \underbrace{\tfrac{1}{n} L \mathbb{1}\mathbb{1}^T}_{=0} = L$$

$$ML = \left(I_n - \tfrac{1}{n}\mathbb{1}\mathbb{1}^T\right)L = L - \underbrace{\tfrac{1}{n}\mathbb{1}\mathbb{1}^T L}_{=0} = L$$

Hence A_c & $(M \otimes I_n)$ commute

Hence the closed-loop dynamics w.r.t. $\delta(t)$ is

$$\dot{\delta}(t) = (M \otimes I_n) A_c x$$

$$= A_c \underbrace{(M \otimes I_n) x}_{= \delta(t)}$$

$$\Rightarrow \boxed{\dot{\delta}(t) = A_c \delta(t)}$$

Sometimes referred to as Disagreement dynamics.

$$\delta = (M \otimes I_n)\, x$$

$$= \begin{bmatrix} x_1 - \frac{1}{n}x_1 - \frac{1}{n}x_2 - \cdots - \frac{1}{n}x_n \\ -\frac{1}{n}x_1 + x_2 - \frac{1}{n}x_2 - \cdots - \frac{1}{n}x_n \\ \vdots \\ -\frac{1}{n}x_1 - \frac{1}{n}x_2 - \cdots + x_n - \frac{1}{n}x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - \frac{1}{n}(x_1 + x_2 + \cdots + x_n) \\ x_2 - \frac{1}{n}(x_1 + x_2 + \cdots + x_n) \\ \vdots \\ x_n - \frac{1}{n}(x_1 + x_2 + \cdots + x_n) \end{bmatrix}$$

Here $\delta_i = 0$ iff $x_i = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$

Hence for $\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{bmatrix} = 0$, we have:

$$\left.\begin{aligned} x_1 &= \frac{1}{n}\left(\sum_{j=1}^{n} x_j\right) \\ x_2 &= \frac{1}{n}\left(\sum_{j=1}^{n} x_j\right) \\ &\vdots \\ x_n &= \frac{1}{n}\left(\sum_{j=1}^{n} x_j\right) \end{aligned}\right\} \rightarrow x_1 = x_2 = \cdots = x_n$$

Consensus is achieved.

- Hence, consensus/synchronization in the n/w can be achieved (asymptotically) by ensuring that $\delta(t) \longrightarrow 0$ as $t \to \infty$.

From the disagreement dynamics

$$\dot{\delta}(t) = A_c \, \delta(t)$$

we need to ensure that the eigenvalues of A_c are in the open left half of the complex plane.

- Let us simplify A_c:

Since L is symmetric, $\exists$ an orthogonal matrix Q s.t.

$$Q^T L Q = \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}}_{\Lambda} \qquad Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$$

Define $\bar{Q} := Q \otimes I_n$

Note $\bar{Q}^T \bar{Q} = (Q^T \otimes I)(Q \otimes I) = I$, Hence

$\bar{Q}$: Orthogonal matrix

Define a new state variable :

$$\epsilon(t) = \bar{Q}^T \delta(t) \quad \Rightarrow \quad \delta(t) = \bar{Q}\,\epsilon(t)$$

closed loop dynamics w.r.t. variable $\delta(t)$

$$\dot{\delta}(t) = A_c\,\delta(t)$$

$$\Rightarrow \bar{Q}\,\dot{\epsilon}(t) = A_c\,\bar{Q}\,\epsilon(t)$$

$$\Rightarrow \dot{\epsilon}(t) = \underbrace{\bar{Q}^T A_c \bar{Q}}_{\tilde{A}_c}\,\epsilon(t)$$

$$\bar{Q}^T A_c \bar{Q} = (Q^T \otimes I_n)\Big((I_n \otimes A) + (-L \otimes \eta BF)\Big)(Q \otimes I_n)$$

$$= \Big((Q^T \otimes A) + (-Q^T L \otimes \eta BF)\Big)(Q \otimes I_n)$$

$$= (Q^T Q \otimes A) + (-\underbrace{Q^T L Q}_{\Lambda_L} \otimes \eta BF)$$

$$= (I_n \otimes A) + (-\Lambda \otimes \eta BF)$$

$$= \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix} + \begin{bmatrix} -\lambda_1 \eta BF & & & \\ & -\lambda_2 \eta BF & & 0 \\ & 0 & \ddots & \\ & & & -\lambda_n \eta BF \end{bmatrix}$$

$$= \begin{bmatrix} A - \lambda_1 \eta BF & & & \\ & A - \lambda_2 \eta BF & & 0 \\ & 0 & \ddots & \\ & & & A - \lambda_n \eta BF \end{bmatrix}$$

Hence $A_c \xrightarrow{\bar{Q}} \tilde{A}_c$

where $\tilde{A}_c$ is a block diagonal matrix.

- Hence, the eigenvalues of A_c are equal to the union of the eigenvalues of $A - \lambda_i \eta BF$

$$\boxed{\text{eig}(A_c) = \bigcup_{i=1}^{n} \text{eig}(A - \lambda_i \eta BF)}$$

Hence, we need to focus on how to place the eigenvalues of $(A - \lambda_i \eta BF)$ in the open-left half of complex plane using the design parameters η & F.

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Recall $\dot{\epsilon}(t) = \tilde{A}_c \epsilon(t)$ $\qquad A_c \xrightarrow{\bar{Q}} \tilde{A}_c$

$$\begin{bmatrix} \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \vdots \\ \dot{\epsilon}_{n-1} \\ \dot{\epsilon}_n \end{bmatrix} = \begin{bmatrix} A-\lambda_1\eta BF & & & & 0 \\ & A-\lambda_2\eta BF & & & \\ & & \ddots & & \\ & 0 & & A-\lambda_{n-1}\eta BF & \\ & & & & A-\lambda_n\eta BF \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_{n-1} \\ \epsilon_n \end{bmatrix}$$

According to the definition

$$\epsilon = (Q^T \otimes I_n)\,\delta \qquad \delta_i \in \mathbb{R}^n, \quad \epsilon_i \in \mathbb{R}^n$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} = \begin{bmatrix} q_{11}\cdot I_n & q_{21}\cdot I_n & \cdots & q_{n1}\cdot I_n \\ q_{12}\cdot I_n & q_{22} I_n & \cdots & q_{n2}\cdot I_n \\ \vdots & \vdots & \ddots & \vdots \\ \boxed{q_{1n}\cdot I_n \quad q_{2n}\cdot I_n \quad \cdots \quad q_{nn}\cdot I_n} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{bmatrix}$$

$$\epsilon_n = (q_n^T \otimes I_n)\,\delta \qquad Q^T = \begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix}$$

q_n is the eigenvector corresponding to eigenvalue $\lambda_n = 0$.

$$\Downarrow$$

$$q_n \in \text{span}\{\mathbb{1}\}$$

Hence q_n can be represented as $q_n = \alpha \cdot \mathbb{1}$

where α is some scalar.

Further, recall $\delta = (M \otimes I_n) x$

Then $\epsilon_n = (q_n^T \otimes I_n) \delta$

$$= (q_n^T \otimes I_n)(M \otimes I_n) x$$

$$= (q_n^T M \otimes I_n) x$$

$$= (\alpha \underbrace{\mathbb{1}^T M}_{=0} \otimes I_n) x$$

$$= 0$$

Hence, $\epsilon_n(t) = 0$ for all $t \geq 0$.

Then we have

$$\boxed{\dot{\epsilon}_i(t) = (A - \lambda_i \eta BF)\, \epsilon(t)} \quad \text{for } i = 1, 2, \cdots n-1$$

Since the n/w graph is connected: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$

Hence, for $i = 1, 2, \cdots n-1$, $\epsilon_i(t)$ will converge to zero asymptotically iff $\mathrm{eig}(A - \lambda_i \eta BF) \in \mathbb{C}^-$.

In $\quad A - \lambda_i \eta BF$

$\underbrace{A, B}_{\text{System Parameters}}$ & $\underbrace{\lambda_i}_{\text{Network topology}}$ are given $\qquad \left.\begin{matrix}\eta \\ F\end{matrix}\right\}$ Design Parameters.

$\rightarrow$ <u>Controller Synthesis</u>

Let us define $\boxed{F := R^{-1} B^T P}$

where P is the symmetric positive definite solution of the following Algebraic Riccati Equation: (ARE)

$$\boxed{A^T P + PA + S - PBR^{-1}B^T P = 0}$$

where S & R are some symmetric positive definite matrices.

- Stability Analysis of : $\dot{\epsilon}_i = (A - \lambda_i \eta BF)\epsilon$

Consider the Lyapunov equation:

$$(A - \lambda_i \eta BF)^T P + P(A - \lambda_i \eta BF)$$

$$= \underbrace{A^T P + PA}_{\text{from ARE}} - \lambda_i \eta F^T B^T P - \lambda_i \eta PBF$$

$$= -S + PBR^{-1}B^T P - \lambda_i \eta PBR^{-1}B^T P - \lambda_i \eta PBR^{-1}B^T P$$

$$= -S - (2\lambda_i \eta - 1)\underbrace{PBR^{-1}B^TP}_{\succ 0}$$

$\underbrace{PBR^{-1}}_{W^T}\underbrace{B^TP}_{W}$

Since $R \succ 0$

$W^TRW \succ 0$

S is SPD (symmetric positive definite)

$PBR^{-1}B^TP$ is SPD

$\lambda_i > 0$ for $i = 1, 2 \cdots n-1$

Let us now choose $\boxed{\eta \geq \dfrac{1}{2\lambda_{n-1}}}$

$$(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0)$$

$$\Rightarrow 2\lambda_i\eta - 1 \geq 0$$

Hence,

$$(A - \lambda_i \eta BF)^T P + P(A - \lambda_i \eta BF) \prec 0$$

(symmetric negative definite

Hence, for the choices of η and F as above

$$\epsilon_i(t) \longrightarrow 0 \quad \text{as} \quad t \to \infty$$

Hence $\epsilon(t) \longrightarrow 0$ as $t \to \infty$

Since $\delta(t) = (Q \otimes I_n)\,\epsilon(t)$

& $Q \otimes I$ is non-singular,

$$\delta(t) \longrightarrow 0 \quad \text{as} \quad t \to \infty$$

$$\Rightarrow x_1 = x_2 = \cdots = x_n \quad \text{as} \quad t \to \infty .$$

Hence consensus is achieved.

→ The closed loop system:

$$\dot{x} = A_c x$$

By defining a variable $\xi = (Q^T \otimes I) x$

we have

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_n \end{bmatrix} = \begin{bmatrix} A - \lambda_1 \eta B F & & & \\ & A - \lambda_2 \eta B F & & \\ & & \ddots & \\ & & & A \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}$$

eigenvalues can not be modified.

Hence if A is not Hurwitz, we can achieve consensus, however, the closed loop system would be unstable.

Hence, we assume the eigenvalues of A are in the open-left half of the complex plane. i.e. each agent is stable.