ELL 805

Lecture-30

Leader - Follower Synchronization :

↳ Follower needs to synchronize with leader.

- Let a MAS has n-follower agents & they are identical. Their dynamics

$$\dot{x}_i = Ax_i + Bu_i \qquad x_i \in \mathbb{R}^n$$
$$u_i \in \mathbb{R}^m$$

- Let the MAS has only one leader, & its dynamics is :

$$\dot{x}_0 = Ax_0 \qquad x_0 \in \mathbb{R}^n$$

x_i : follower state trajectory

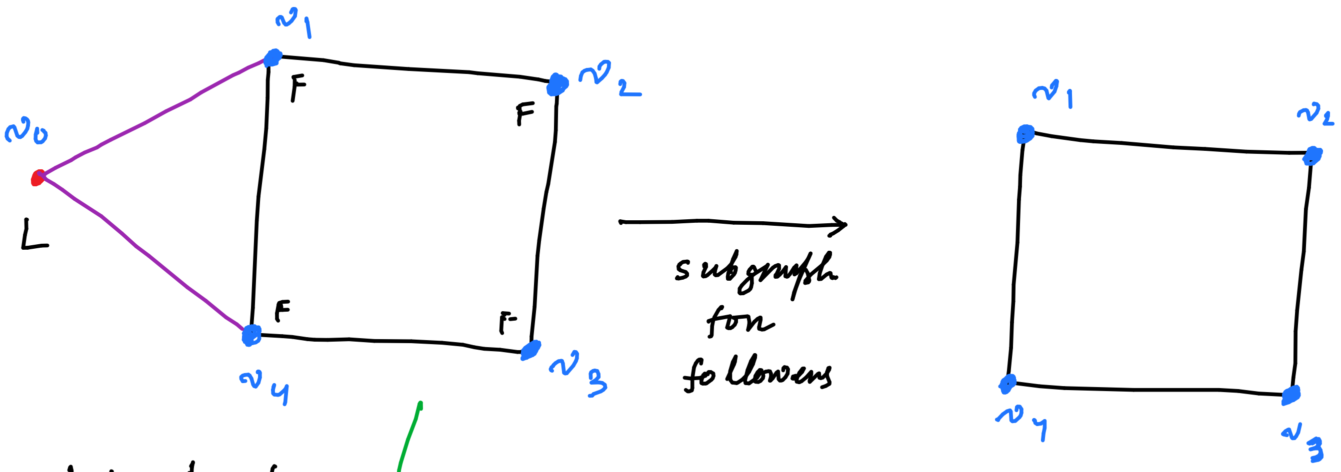
x_0 : Leader state trajectory

Objective : Design u_i for the followers s.t. the follower states $x_i(t)$ will synchronize with the leader state $x_0(t)$.

i.e. $\boxed{\lim_{t \to \infty} x_i(t) - x_0(t) = 0}$

- Assume that the n/w graph is connected & there is at least one follower with which the leader is communicating.

→ The n/w graph representation for Leader & follower:



L : leader
F : Follower

Laplacian matrix

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

$$\mathcal{L}_g = \underbrace{\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}}_{\mathcal{L}} + \underbrace{\begin{bmatrix} g_1 & & & \\ & 0 & & \\ & & 0 & \\ & & & g_4 \end{bmatrix}}_{G}$$

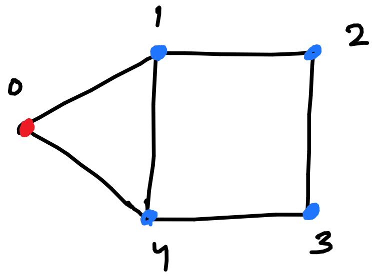
↳ Called Pinning matrix which is a diagonal matrix with diagonal entries $g_i > 0$, if the leader is communicating with i^{th} follower agent; otherwise 0.

$$\boxed{\mathcal{L}_g := \mathcal{L} + G}$$

• Define the relative state information available at follower agents:

$$\boxed{z_i = \sum_{j \in \mathcal{N}(i)} w_{ij}(x_j - x_i) + g_i(x_0 - x_i)}$$

$\mathcal{N}(i)$: set of neighborhood agents of ith agent.



for $x_i \in \mathbb{R}$

$$\left\{\begin{aligned}
z_1 &= (x_4 - x_1) + (x_2 - x_1) + g_1(x_0 - x_4)\\
z_2 &= (x_3 - x_2) + (x_1 - x_2)\\
z_3 &= (x_2 - x_3) + (x_4 - x_3)\\
z_4 &= (x_1 - x_4) + (x_3 - x_4) + g_4(x_0 - x_4)
\end{aligned}\right.$$

$$\underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}}_{\bar{z}} = \underbrace{\begin{bmatrix} -(2+g_1) & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -(2+g_4) \end{bmatrix}}_{\substack{-(L+G)\\ = -L_g}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix} g_1 & & & \\ & 0 & & \\ & & 0 & \\ & & & g_4 \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} x_0 \\ x_0 \\ x_0 \\ x_0 \end{bmatrix}}_{\bar{x}_0}$$

then $z = -(L+G)x + Gx_0$

$$\Rightarrow \boxed{z = -(L+G)x + (L+G)x_0}$$

Note

$Gx_0 = (L+G)x_0$

Since $Lx_0 = 0$

For $x_i \in \mathbb{R}^n$

$$\boxed{z = (-L_g \otimes I_n) x + (L_g \otimes I_n) x_0}$$

Let the control signal be (for i^{th} agent)

$$\boxed{u_i = \eta F z_i}$$

where $\eta \in \mathbb{R}$ & $F \in \mathbb{R}^{m \times n}$ are design parameters.

$$\boxed{u = (I_n \otimes \eta F) z}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$$\dot{x}_i = A x_i + B u_i$$

$$\dot{x} = (I_n \otimes A) x + (I_n \otimes B) u$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\dot{x} = (I_n \otimes A) x + (I_n \otimes B)(I_n \otimes \eta F) z$$

$$= (I_n \otimes A) x + (I_n \otimes \eta B F)\left[(-L_g \otimes I_n) x + (L_g \otimes I_n) x_0\right]$$

$$= (I_n \otimes A) x + (-L_g \otimes \eta B F) x + (L_g \otimes \eta B F) x_0$$

$$= \Big[\underbrace{(I_n \otimes A) + (-L_g \otimes \eta B F)}_{A_c}\Big] x + (L_g \otimes \eta B F) x_0$$

For the leader agents

$$x_0 = A x_0$$

$$\begin{bmatrix} x_0 \\ x_0 \\ \vdots \\ x_0 \end{bmatrix} = \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix} \begin{bmatrix} x_0 \\ x_0 \\ \vdots \\ x_0 \end{bmatrix}$$

$$X_0 = (I_n \otimes A) X_0$$

Define an error vector $\delta(t) = X(t) - X_0(t)$

Then $$\dot{\delta} = \dot{X} - \dot{X}_0$$

$$\Rightarrow \quad \dot{\delta} = A_c X + (L_g \otimes \eta BF) X_0 - (I_n \otimes A) X_0$$

$$= A_c X - \Big[\underbrace{(I_n \otimes A) + (-L_g \otimes \eta BF)}_{A_c} \Big] X_0$$

$$= A_c (X - X_0)$$

$$= A_c \delta$$

Hence $\boxed{\dot{\delta} = A_c \delta}$ ← Synchronization error dynamics

$\downarrow$

Synchronization can be achieved iff $\delta \to 0$ as $t \to \infty$ i.e. $x_i \to x_0$ as $t \to \infty$.

Lecture - 31

- Where are the eigenvalues of matrix L_g ?

  Is there any zero eigenvalues of L_g ?

→ <u>Some more results</u>:

- A matrix $M \in \mathbb{R}^{n\times n}$ is called "<u>diagonally dominant</u>" if

  $$\boxed{|m_{ii}| \geq r_i} \quad \text{where} \quad R_i = \sum_{\substack{j=1\\ i\neq j}}^{n} |m_{ij}| \, ,$$

  and it is called "<u>strictly diagonally dominant</u>" if $\boxed{|m_{ii}| > r_i}$.

- The digraph representation of a matrix $M \in \mathbb{R}^n$ is obtained as follows:

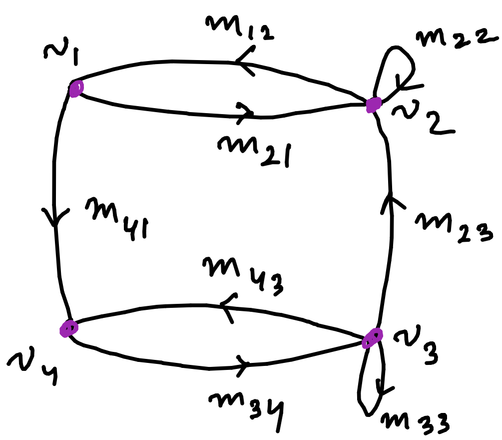
  (i) The digraph has 'n' vertices (equal to the no. of rows or colums of M)

  The vertices are $\{v_1, v_2 \dots v_n\}$.

(ii) There is an edge from v_j to v_i if $m_{ij} \neq 0$.

For $$M = \begin{bmatrix} 0 & m_{12} & 0 & 0 \\ m_{21} & m_{22} & m_{23} & 0 \\ 0 & 0 & m_{33} & m_{34} \\ m_{41} & 0 & m_{43} & 0 \end{bmatrix}$$

Digraph representation of M



- **Strongly Connected Property of M**

A matrix $M \in \mathbb{R}^{n \times n}$ has the "strongly connected (SC) property if the associated digraph is strongly connected.

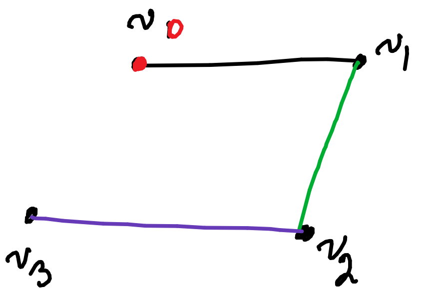
→ <u>Result</u>

Assume that $M \in \mathbb{R}^{n\times n}$ has the SC property. If M is diagonally dominant, and if $|m_{ii}| > r_i$ for <u>at least</u> one value of $i = 1, 2 \cdots n$, then the matrix M is invertible.

[Ref: R. A. Horn & C. R. Johnson, "Matrix Analysis", Cambridge University Press, 1990]

→ We have assumed that the network graph G is connected.

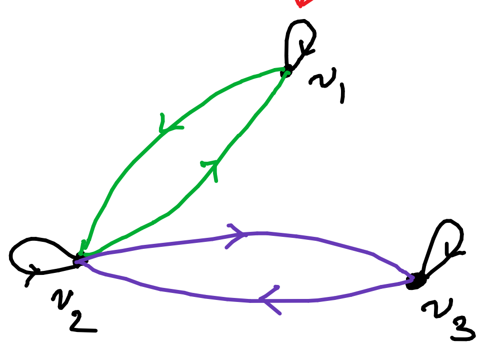
The digraph representation of L_g:



network graph

$$\rightarrow \quad L_g = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \begin{bmatrix} g_1 & & \\ & 0 & \\ & & 0 \end{bmatrix}}$$

Digraph of L_g



Each edge between the followers, is replaced by two directed edges with opposite direction.

- Since $\mathcal{L}_g$ is symmetric, and n/w graph is connected, we will get a strongly connected digraph associated with $\mathcal{L}_g$.

$$\Downarrow$$

$\mathcal{L}_g$ has SC property.

→ $\mathcal{L}_g$ is diagonally dominant, and it satisfies $|l_{g_{ii}}| > r_i$ for at least one $i = 1, 2 \cdots n$.

Due to the presence of $g_i > 0$.

$$\Downarrow$$

$\mathcal{L}_g$ is invertible, & hence, there are No zero eigenvalues of $\mathcal{L}_g$.

→ For the synchronization error dynamics:

$$\dot{\delta} = A_c \delta$$

Let $\epsilon(t) = (Q^T \otimes I_n)\,\delta$ where $Q^T \mathcal{L}_g Q = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

$$\Lambda$$

→ Similar to the consensus problem, we have

$$\dot{\epsilon}(t) = \left[(Q^T \otimes I_n) A_c (Q^{-1} \otimes I_n)\right] \epsilon(t)$$

$$= \left[(I_n \otimes A) + (-\Lambda \otimes \eta BF)\right] \epsilon(t)$$

$\Downarrow$

$$\boxed{\dot{\epsilon}_i(t) = (A - \lambda_i \eta BF)\epsilon(t)}$$ ← Decoupled error dynamics.

↓

We have to ensure that the eigenvalues of $A_c = \bigcup_{i=1}^{n} \text{eig}(A - \lambda_i \eta BF)$ are in $\mathbb{C}^-$.

→ def $\boxed{F = R^{-1} B^T P}$

where P is the unique symmetric positive definite solⁿ of A.R.E :

$$\boxed{A^T P + PA + S - PBR^{-1}B^T P = 0}$$

S & R are some symmetric positive definite matrices.

Further choose $\boxed{\eta \geq \frac{1}{2\lambda_2}}$

λ_2 is the smallest eigenvalue of L_g.

- Then we had shown in Consensus Problem that the eigenvalues of $(A - \lambda_i \eta BF)$ cluster within the open left half of Complex plane $(\mathbb{C}^-)$.

$$\Downarrow$$

$$\epsilon(t) \longrightarrow 0 \quad \text{as } t \to \infty$$

Since $Q \otimes I_n$ is orthogonal,

$$\delta(t) = (Q \otimes I)\, \epsilon(t) \longrightarrow 0 \quad \text{as } t \to \infty.$$

Hence $\quad x_i(t) \longrightarrow x_0(t) \quad$ as $t \to \infty$.

Hence, leader-follower synchronization objective achieved.

→ Under the assumptions on agent dynamics that they are <u>controllable</u> and <u>observable</u>, one can always design F for achieving synchronization.