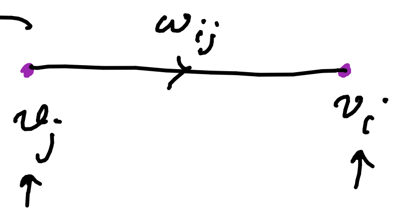
ELL 805

Lecture - 3

Directed Graph

edge is denoted as (v_j, v_i)

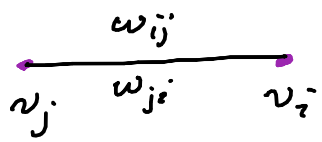


Initial vertex

Final vertex

$w_{ij} \neq w_{ji}$

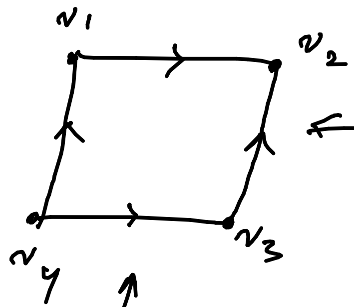
Undirected Graph



$w_{ij} = w_{ji}$

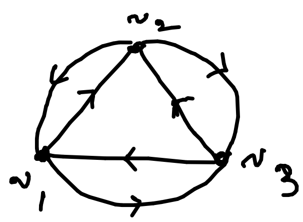
→ **Strongly Connected Digraph:**

For <u>every</u> pair of vertices v_i & v_j in the directed graph $G_d(V, E)$, if there exists a <u>directed path</u>, then the digraph is strongly connected.



← Not strongly connected.

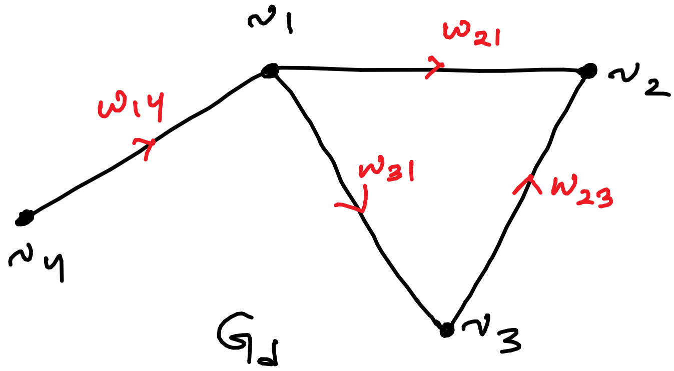
Weakly connected



strongly connected graph

- Weakly Connected Digraph:

A digraph is weakly connected if it is connected, when it is viewed as an undirected graph.



$\mathcal{N}_I(1) = \{4\}$
$\mathcal{N}_I(2) = \{1, 3\}$
$\mathcal{N}_O(1) = \{2, 3\}$
$\mathcal{N}_O(2) = \{\ \}$, $\mathcal{N}_O(3) = \{2\}$

- Define the following sets:

$$\mathcal{N}_I(i) := \left\{ j : \text{there is an edge in } G_d \text{ that leaves from } v_j \text{ and enters at } v_i \right\}$$

$$\mathcal{N}_O(i) := \left\{ j : \text{there is an edge in } G_d \text{ that leaves from } v_i \text{ and enters at } v_j \right\}$$

- <u>In-degree of a vertex in a Digraph</u>

$$\boxed{d_{in}(v_i) := \sum_{j \in \mathcal{N}_I(i)} w_{ij}}$$

The addition of the weights of the edges which <u>enters</u> to v_i.

In above figure $\begin{cases} d_{in}(v_1) = w_{14} \\ d_{in}(v_2) = w_{21} + w_{23} \end{cases}$ $\quad d_{in}(v_3) = w_{31}$

- **Out-degree of a vertex in a digraph**

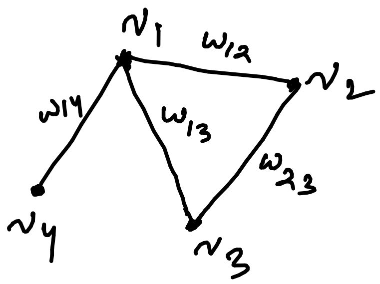
The addition of edge weights, which leave from a vertex v_i

$$\boxed{d_{out}(v_i) = \sum_{j \in \mathcal{N}_o(i)} w_{ji}}$$

$d_{out}(v_1) = w_{21} + w_{31}$ $\quad$ $d_{out}(v_2) = 0$

$d_{out}(v_3) = w_{23}$

→ **Undirected graph (G)**



$w_{ij} = w_{ji}$

Degree of a vertex

$$d(v_i) = \sum_{j \in \mathcal{N}(i)} w_{ij}$$

← neighborhood set

$$d(v_1) = w_{12} + w_{13} + w_{14}$$
$$= w_{12} + w_{31} + w_{41}$$

→ **Assume the Edge weights $w_{ij} = 1$**

$$w_{ij} = 1$$

- For Undirected graph:

Degree of a vertex v_1 $\quad d(v_1) = 3$

↑ $\quad d(v_4) = 1$

Cardinality of $\mathcal{N}(i)$ $\quad d(v_3) = 2$

Neighborhood set $\mathcal{N}(i) = \{ j \mid v_i \text{ \& } v_j \text{ are adjacent} \}$

on the no. of edges incident to vertex v_i

- For directed graph, if $w_{ij} = 1$

in-degree of a vertex $= |\mathcal{N}_I(i)|$

out-degree of a vertex $= |\mathcal{N}_O(i)| \leftarrow$ cardinality

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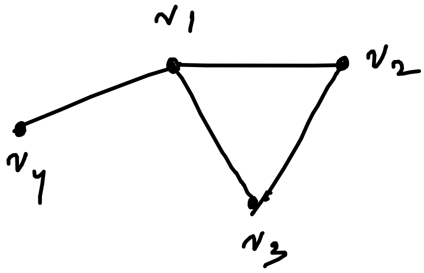
→ Matrices associated with Undirected graphs

- Adjacency Matrix :

$A(G)$

$$[A(G)]_{ij} = \begin{cases} 1, & \text{if } v_i \text{ \& } v_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}$$



$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- We will assume that there are No self-loops in a graph.

⇓

the diagonal elements of $A(G)$ are zero.

- $A(G)$ is a symmetric matrix.

- <u>Degree matrix of G</u>

It is a Diagonal matrix with diagonal entries are the degree of the vertices. $d(v_i)$

Degree matrix $\Delta = \begin{bmatrix} d(v_1) & & & 0 \\ & d(v_2) & & \\ & & \ddots & \\ 0 & & & d(v_n) \end{bmatrix}$

$$\Delta = \begin{bmatrix} 3 & & & 0 \\ & 2 & & \\ & & 2 & \\ 0 & & & 1 \end{bmatrix}$$

→ Laplacian Matrix of G

Laplacian matrix → $\mathcal{L} = \Delta - A(G)$

$$\mathcal{L} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

- For an undirected graph the Laplacian matrix is symmetric.
- Row sum and column sum are zero for every rows & colums.