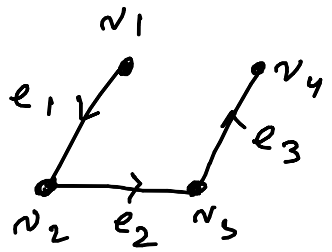
ELL 805

Lecture - 4

Incidence Matrix D : For a given graph (G) with some orientation of the edges, the incidence matrix D is of size $n \times m$, where 'n' is the no. of vertices & m is the no. of edges in G, whose elements are :

If the graph is undirected, then just randomly assign some direction to its some edges e_j & then construct D.

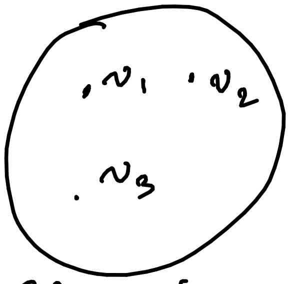
$$[D]_{ij} = \begin{cases} -1 & \text{if } v_i \text{ is initial vertex of } e_j \\ & (e_j \text{ leaves from } v_i) \\ +1 & \text{if } v_i \text{ is final vertex of } e_j \\ 0 & \text{otherwise.} \end{cases}$$



$$D = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}_{4 \times 3}$$

$n = 4$ no. of vertices

$m = 3$ no. of edges



$\mathcal{V}$ set

$v_i \in \mathbb{R}^n$

$$\begin{bmatrix} v_{1_1} \\ v_{2_1} \\ \vdots \\ v_{n_1} \end{bmatrix}$$

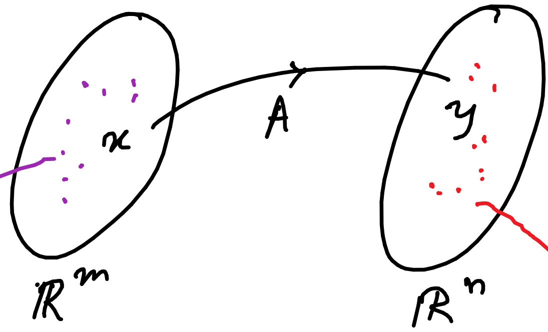
Basis : linearly independent vectors in $\mathcal{V}$.

Dimension of a vector space :

$\dim(\mathcal{V})$ = The number of basis in $\mathcal{V}$ is called dimension.

Two subspaces of $\mathbb{R}^n$

$A \in \mathbb{R}^{n \times m}$ $\qquad$ $\left[\quad A \quad \right]_{n \times m}$



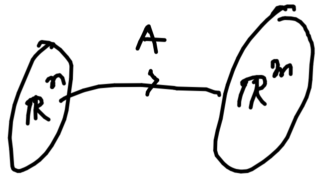
$y = Ax$

Image space of A : $Im(A) := \{ y \in \mathbb{R}^n \mid Ax = y \}$

$\left.\begin{array}{l}\text{Kernel space of } A \\ \text{Null space of } A\end{array}\right\}$ $\mathcal{N}(A) := \{ x \in \mathbb{R}^m \mid Ax = 0 \}$

→ Rank-Nullity Theorem :

For a given matrix $A \in \mathbb{R}^{m \times n}$



$$\boxed{\text{rank}(A) + \text{Nullity}(A) = n}$$

dimension of Null space of matrix A.

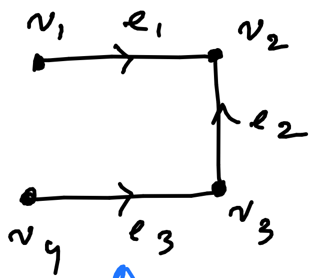
——— x ———

- **Result** : Consider a connected (undirected) Graph G on 'n' vertices with 'm' edges. Then, we have

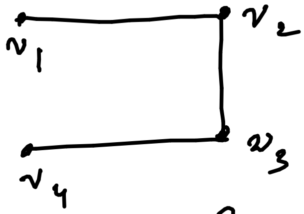
$D \in \mathbb{R}^{n \times m}$

$$\boxed{\text{rank}(D) = n - 1}$$

Incidence matrix of G



↑ Constructed by assigning arbitrary orientation to the edges.



Given G

$$D = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \end{array}$$

Properties of D

- Colum sum is zero
- Each colum has maximum two non-zero entries : 1, -1.

$$D^T = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x}$$

- Let us look at the vectors x which belongs to the $\mathcal{N}(D^T)$.
$(D^T x = 0)$

- All elements of x for the above example need to be <u>same</u> to satisfy $D^T x = 0$

Since each row of D^T has two non-zero elements : -1 & $+1$, every pair of x_i & x_j (elements of x) must be <u>equal</u> to satisfy $D^T x = 0$.

$$x_i = x_j$$

- If the graph has an isolated vertex v_j then the jth row of D is zero, which is equivalent to saying that jth colum of D^T is zero.

In such case x_j component, one can choose any real number; provided all other entries of x are equal.

- Since we have assumed that the graph G is connected.

$\Downarrow$

There are no isolated vertex in G

$\Downarrow$

None of the rows of D is completely zero.

$\Downarrow$

$$D^T x = 0 \rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} \right.$$

$\Downarrow$

There is only one linearly independent vector x in the Null space of D^T.

$$\dim(\mathcal{N}(D^T)) = 1$$

$D \in \mathbb{R}^{n \times m}$

$$D^T \in \mathbb{R}^{m \times n}$$

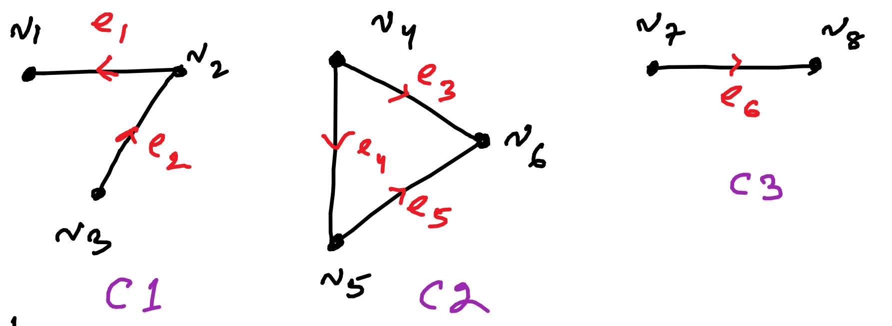
According to Rank - Nullity Theorem

$$\text{rank}(D^T) + \underbrace{\text{nullity}(D^T)}_{= 1} = n$$

$\text{rank}(A) = \text{rank}(A^T)$

$$\Rightarrow \text{rank}(D^T) = n-1 \Rightarrow \text{rank}(D) = \text{rank}(D^T) = n-1.$$

• Consider an undirected disconnected graph



Disconnected graph with 3-connected components (Ci)

$$D = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \left[\begin{array}{cc:ccc:c} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

↑ Block diagonal matrix

$$D = \begin{bmatrix} D_{c1} & & 0 \\ & D_{c2} & \\ 0 & & D_{c3} \end{bmatrix}$$

D_{ci} refers to the incidence matrix corresponding to the ith connected component.