ELL 805

Lecture - 5

For a disconnected graph, the incidence matrix

$$D = \begin{bmatrix} D_{c1} & & & 0 \\ & D_{c2} & & \\ & 0 & \ddots & \\ & & & D_{ck} \end{bmatrix}$$

D_{ci}'s are the incidence matrices of ci-connected component.

→ **Result** : Consider a disconnected graph G on 'n' vertices with 'k' connected components. Then the $\boxed{\text{rank}(D) = n - k}$. where D is the incidence matrix of G.

Proof Assume that each connected component has n_i number of vertices.

$$\Rightarrow n_1 + n_2 + \cdots + n_k = n$$

$$D = \begin{bmatrix} D_{c1} & & & \\ & D_{c2} & & \\ & & \ddots & \\ & & & D_{ck} \end{bmatrix}$$

D_{ci} - corresponds to the ci. connected component's incidence matrix.

Since c_i has n_i vertices, according to the previous result:

$$\text{rank}(D_{c_i}) = n_i - 1$$

$$\begin{aligned}\text{rank}(D) &= \text{rank}(D_{c_1}) + \text{rank}(D_{c_2}) + \cdots + \text{rank}(D_{c_k}) \\ &= (n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) \\ &= (n_1 + n_2 + \cdots + n_k) - k \\ &= n - k.\end{aligned}$$

$\boxtimes$

The dimension of $\mathcal{N}(D) = k$.

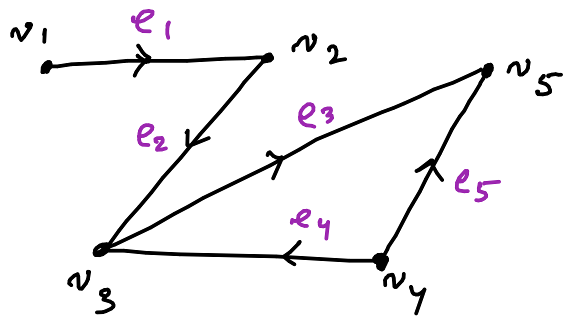
→ <u>Signed Path Vector(z)</u>

Given a directed graph G_d, a signed path vector (z) corresponding to a path in G_d, which has following entries:

$$z = [z_1 \; z_2 \; \cdots \; z_m]^T$$

m: Number of edges in G_d.

$$z_i = \begin{cases} 1, & \text{if the edge } e_i \text{ traverse positively (in the direction of path)} \\ -1, & \text{if } e_i \text{ traverse -vely (in the reverse direct}^n \text{ of path)} \\ 0, & \text{otherwise.} \end{cases}$$



- Let us consider a path from v_1 to v_4 with edges e_1, e_2 & e_4.

$$z = [1 \;\; 1 \;\; 0 \;\; -1 \;\; 0]^T$$

$v_1 \rightarrow v_4$ with edges e_1, e_2, e_3 & e_5.

$$z = [1 \;\; 1 \;\; 1 \;\; 0 \;\; -1]^T$$

→ **Result**: Given a directed path with distinct initial & final vertices, described by a signed path vector 'z', in the digraph G_d, the vector $y = D(G_d)z$ has the following entries

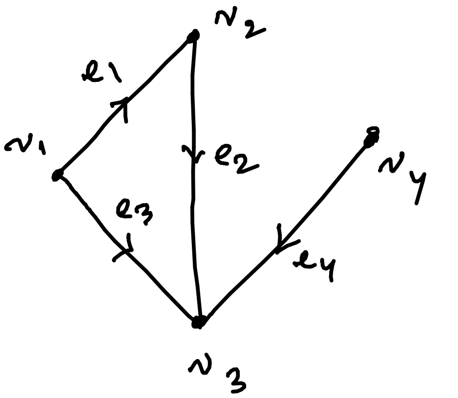
incidence matrix of G_d

$$y_i = \begin{cases} -1, & \text{if the vertex } v_i \text{ is the initial vertex of the path} \\ 1, & \text{if the vertex } v_i \text{ is the final vertex of the path} \\ 0, & \text{otherwise} \end{cases}$$

Proof

$$y = D(G_d)\, x$$

$$= \underbrace{D(G_d) \underbrace{\begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_m \end{bmatrix}}_{diag(x)}}_{D\cdot diag(x)} \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbb{1}}$$



Consider a directed path from v_1 to v_4 with edges e_1, e_2, e_4

$$x = \begin{bmatrix} 1 & 1 & 0 & -1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & +1 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & 0 & \\ & & 0 & \\ 0 & & & -1 \end{bmatrix}}_{diag(x)}$$

$$= \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is a kind of incidence matrix, corresponding to the directed path we have considered.

- In the matrix $D(G_d)\, diag(x)$, the colums of $D(G_d)$ will appear with appropriate scaling done by the elements of x.
   - The edge that does not appear in a directed path, the corresponding colum of $D\cdot diag(x)$ is '0'.

$$\underline{D(G_d)\, x \;=\; \underbrace{D(G_d)\cdot \text{diag}(x)\cdot \mathbb{1}}}$$

the addition of all elements in a row of $D(G_d)\cdot \text{diag}(x)$.

$\underbrace{D(G_d)\cdot \text{diag}(x)}_{D_x}$,

- If vertex v_i is the initial vertex of the path, then ith row of D_x has only one non-zero element : -1.

  Then the ith row addition $= -1$

- If vertex v_i is the final vertex of the path, then ith row of D_x has only one non-zero element : 1.

  $\Downarrow$

  The addition of the ith row $= 1$

- If the vertex v_i, which appear in between the initial & final vertices of the directed path, then ith row of D_x has two non-zero elements : $\underline{1}$, $\underline{-1}$ Since one edge enters at v_i & one edge leaves from v_i

  $\Downarrow$

  then the addition of ith row $= 0$

$y = D_z \cdot \mathbb{1}$ (row addition of D_z)

$\Downarrow$

y_i has entries either $1, -1$ or 0. ☒

- If a we have a directed path with same initial & final vertices (cycle) then at every vertex v_i, there will be one edge which enters, and one edge, which leaves.

$\Downarrow$

At every row of D_z, there will be two non-zero element (according to the previous proof) : 1 & -1.

$\Downarrow$

$$D_z \cdot \mathbb{1} = 0$$

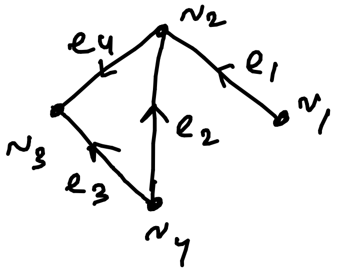
$\Rightarrow$ Every element of $y = D(G_d) z$ is zero.

$\Downarrow$

$$D(G_d) z = 0$$

$\Downarrow$

$$z \in \mathcal{N}(D(G_d))$$



$$\underbrace{\begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix} \end{array}}_{D(G_d)} \underbrace{\begin{bmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}}_{\mathrm{diag}(z)}$$

cycle consisting of edges e_2, e_4 & e_3

$\downarrow$ corresponding

$$z = \begin{bmatrix} 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow D_z = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$D_z \cdot \mathbb{1} = 0$$

$\Rightarrow$ $D(G_d) z = 0$ $\Rightarrow$ z is in the null space of $D(G_d)$.

- Hence, the vector z that corresponds to a cycle, belongs to the null space of $D(G_d)$.

- Cycle space : The null space of incidence matrix $D(G_d)$ of a digraph G_d is called "cycle space" of G_d.

- Result : Given a directed graph G_d with incidence matrix $D(G_d)$, the null space of $D(G_d)$ is spanned by all linearly independent signed path vectors z, which correspond to the cycles of G_d.