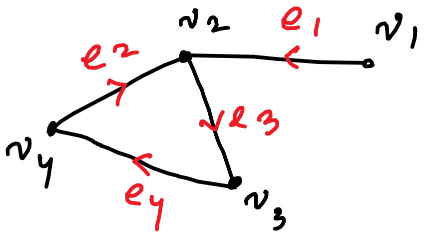
ELL 805

Lecture - 6



Incidence matrix

$$D = \begin{array}{c|cccc} & e_1 & e_2 & e_3 & e_4 \\ \hline v_1 & & & & \\ v_2 & & & & \\ v_3 & & & & \\ v_4 & & & & \end{array} \quad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$D^T = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline e_1 & & & & \\ e_2 & & & & \\ e_3 & & & & \\ e_4 & & & & \end{array} \quad \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

For any matrix

$A \to A(i, :)$

$\uparrow$ i^{th} row of A

$A \to A(:, j)$

$\uparrow$ j^{th} colm of A

$D(i, :)\, D^T(:, i)$

$$D(2,:)\, D^T(:,2) = \begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = 3$$

(No. of edges incident to v_2, if we see the weakly connected version)

$$D(1,:)\, D^T(:,1) = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1$$

No of edges incident to v_1

$$D(3,:)\, D^T(:,3) = \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = 2$$

$\leftarrow$ No of edges incident to v_3

$D(i,:)D^T(:,i)$ = The no. of edges incident to vertex v_i

= The cardinality of neighborhood set of v_i $(\mathcal{N}(i))$

= The <u>degree of vertex</u> v_i

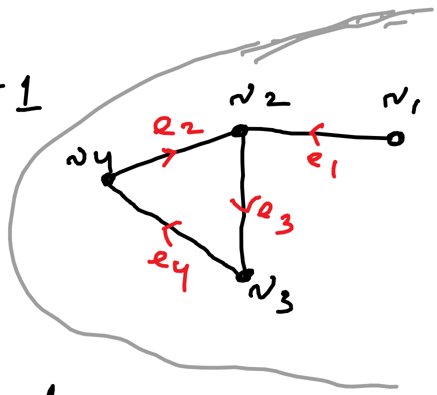
The graph is considered as an undirected graph.

$D(i,:)D^T(:,j)$

e_1 entering at v_2

$$D(1,:)D^T(:,2) = \underbrace{\begin{bmatrix}-1 & 0 & 0 & 0\end{bmatrix}}_{v_1} \underbrace{\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}}_{v_2} = -1$$

e_1 leaving from v_1



$$D(2,:)D^T(:,3) = \underbrace{\begin{bmatrix}1 & 1 & -1 & 0\end{bmatrix}}_{v_2} \underbrace{\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}}_{v_3} = -1$$

e_3 is leaving from v_2

e_3 is entering at v_3

$$D(1,:)D^T(:,4) = \begin{bmatrix}-1 & 0 & 0 & 0\end{bmatrix} \begin{bmatrix}0\\-1\\0\\1\end{bmatrix} = 0$$

$$D(1,:)D^T(:,3) = \underbrace{\begin{bmatrix}-1 & 0 & 0 & 0\end{bmatrix}}_{v_1} \underbrace{\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}}_{v_3} = 0$$

$$D(i,:)D^T(:,j) = \begin{cases} -1, & \text{if the vertices } v_i \text{ \& } v_j \text{ are adjacent} \\ 0, & \text{if } v_i \text{ \& } v_j \text{ are not adjacent.} \end{cases}$$

$i \neq j$

- Laplacian Matrix

$$\boxed{\mathcal{L} := \underset{\substack{\uparrow \\ \text{degree matrix}}}{\Delta} - A}$$ Adjacency matrix

$$\mathcal{L} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathcal{L}_{ii} = D(i,:)\, D^T(:,i)$$

$$\mathcal{L}_{ij} = D(i,:)\, D^T(:,j) \qquad i \neq j$$

$$\boxed{\mathcal{L} := D D^T}$$ D : incidence matrix

— x —

$A \in \mathbb{R}^{n\times n}$ $\qquad A = A^T$

- Symmetric positive semidefinite:

For <u>every</u> $x \in \mathbb{R}^n$

$$x^T A x \geq 0$$

- symmetric positive definite matrix:

For <u>every</u> $x \in \mathbb{R}^n$

$$x^T A x > 0$$

- For any $A \in \mathbb{R}^{n \times m}$: $\underbrace{A^T A}$ or $\underbrace{A A^T}$ symmetric matrix

- Laplacian matrix $L = DD^T \Rightarrow L$ is symmetric

Let $x \in \mathbb{R}^n$ $\qquad$ $L \in \mathbb{R}^{n \times n}$ when n is the no of vertices in G.

$$x^T L x = \underbrace{x^T D}_{y^T} \underbrace{D^T x}_{y}$$

$$= y^T y = y_1^2 + y_2^2 + \cdots + y_n^2$$

Since D is rank deficient & also we had seen that $\dim(\mathcal{N}(D^T)) = 1$

$\Downarrow$

If we choose $x \in \mathcal{N}(D^T)$

$\Downarrow$

$D^T x = y = 0$ for $x \neq 0$

$\Downarrow$

One can find a non-zero $x \in \mathbb{R}^n$ s.t.

$$y = D^T x = 0$$

$$\Rightarrow x^T D D^T x = \sum_{i=1}^{n} y_i = 0$$

If $x \notin \mathcal{N}(D^T) \Rightarrow y \neq 0 \Rightarrow \Sigma y_i \neq 0$

$$x^T L x = x^T D D^T x \geq 0$$

$$\Downarrow$$

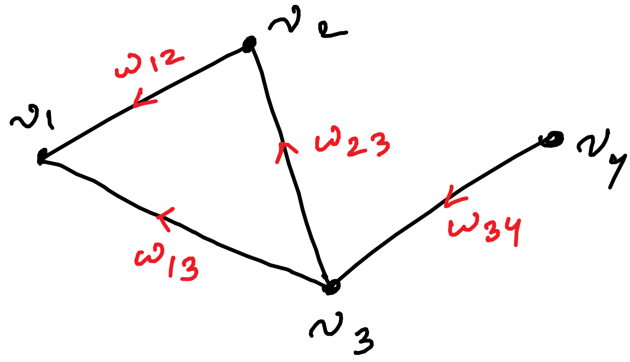
- Laplacian matrix L for an undirected graph is symmetric positive semidefinite matrix.

- For directed Graphs with Edge weights (w_{ij})
  (Unidirectional communication) between the agents.

- Weighted Adjacency matrix

  The entries

$$[A_w(G_d)]_{ij} = \begin{cases} w_{ij}, & \text{if there is an edge from vertex } v_j \text{ to } v_i \\ 0, & \text{otherwise.} \end{cases}$$



Weighted Adjacency matrix

$$A_w(G_d) = \begin{bmatrix} 0 & w_{12} & w_{13} & 0 \\ 0 & 0 & w_{23} & 0 \\ 0 & 0 & 0 & w_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$