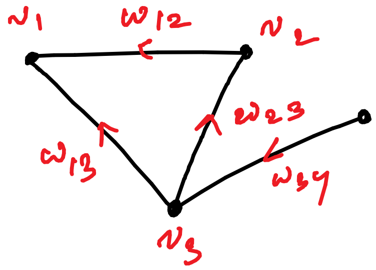
ELL805

Lecture-7

- **Weighted Directed Graphs (Digraphs)**



$$A_w(G) = \begin{bmatrix} 0 & w_{12} & w_{13} & 0 \\ 0 & 0 & w_{23} & 0 \\ 0 & 0 & 0 & w_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Weighted Adjacency matrix.

- **Weighted In-degree of a vertex v_i**

$$d_{in}(v_i) = \sum_{j \in \mathcal{N}_I(i)} w_{ij}$$

weighted digraph

$$\mathcal{N}_I(i) := \left\{ j \; : \; \text{there is an edge in } G_d \text{ that leaves from } v_j \text{ and enters at } v_i. \right\}$$

$$d_{in}(v_1) = w_{13} + w_{12}$$

$$d_{in}(v_2) = w_{23}$$

- **Weighted In-degree Matrix:**

$$\Delta_w(G_d) = \begin{bmatrix} d_{in}(v_1) & & & \mathbf{0} \\ & d_{in}(v_2) & & \\ & & \ddots & \\ \mathbf{0} & & & d_{in}(v_n) \end{bmatrix}$$

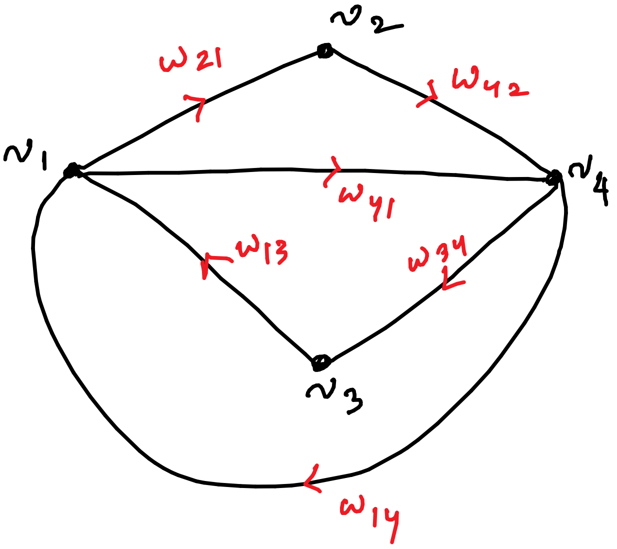
- Weighted In-degree Laplacian Matrix

$$L_w(G_d) = \Delta_w(G_d) - A_w(G_d)$$

$$L_w(G_d) = \begin{bmatrix} w_{12}+w_{13} & -w_{12} & -w_{13} & 0 \\ 0 & w_{23} & -w_{23} & 0 \\ 0 & 0 & w_{34} & -w_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- L_w is a square matrix
- L_w is not symmetric
- The row sum of L_w is zero, whereas the column sum is not zero.

→ <u>Balanced Weighted Digraph</u>



G_d

- A weighted digraph is a balanced weighted digraph if the in-degree = out degree at every vertices of G_d

- $d_{in}(v_1) = w_{13} + w_{14}$ $\quad$ $d_{out}(v_1) = w_{21} + w_{41}$
- $d_{in}(v_2) = w_{21}$ $\quad$ $d_{out}(v_2) = w_{42}$

$d_{in}(v_3) = w_{34}$ $\qquad$ $d_{out}(v_3) = w_{13}$

$d_{in}(v_4) = w_{41} + w_{42}$ $\qquad$ $d_{out}(v_4) = w_{14} + w_{34}$

For balanced digraph

$$\boxed{d_{in}(v_i) = d_{out}(v_i)}$$

$d_{in}(v_4) = d_{out}(v_4) \Rightarrow w_{41} + w_{42} = w_{14} + w_{34}$

$d_{in}(v_3) = d_{out}(v_3) \Rightarrow w_{34} = w_{13}$

- Weighted Adjacency Matrix

$$A_w(G_2) = \begin{bmatrix} 0 & 0 & w_{13} & w_{14} \\ w_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & w_{34} \\ w_{41} & w_{42} & 0 & 0 \end{bmatrix}$$

Not symmetric.

Indegree Matrix

$$\Delta_w(G_d) = \begin{bmatrix} w_{13}+w_{14} & & & \\ & w_{21} & & 0 \\ & 0 & w_{34} & \\ & & & w_{42}+w_{41} \end{bmatrix}$$

$$L_w(G_d) = \begin{bmatrix} w_{13}+w_{14} & 0 & -w_{13} & -w_{14} \\ -w_{21} & w_{21} & 0 & 0 \\ 0 & 0 & w_{34} & -w_{34} \\ -w_{41} & -w_{42} & 0 & w_{41}+w_{42} \end{bmatrix}$$

- Not a symmetric matrix

- For a balanced weighted digraph, since

$$d_{in}(v_i) = d_{out}(v_i)$$

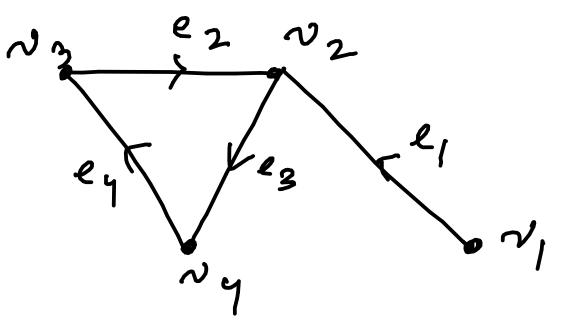
the Laplacian Matrix L_w has
  - ↳ Row sum equal to zero
  - ↳ Column sum equal to zero.

- <u>Undirected Graph (with edge weight = 1)</u>

Laplacian Matrix
$$\boxed{\begin{aligned} L &= \Delta - A \\ &= DD^T \end{aligned}}$$

A : Adjacency matrix
Δ : Degree matrix
D : Incidence Mat

- Edge Laplacian Matrix : $\boxed{L_e = D^T D}$



$$D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$D^T = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Corresponds to e_3.

$$D^T(i,:)\,D(:,i)$$

$$D^T(1,:)\,D(:,1) = \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 2$$

$$D^T(3,:)D(:,3) = \underbrace{\begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix}}_{e_3} \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{e_3} = 2$$

$$D^T(4,:)D(:,4) = \underbrace{\begin{bmatrix} 0 & 0 & -1 & 1 \end{bmatrix}}_{e_4} \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{e_4} = 2$$

$$\boxed{L_e(i,i) = D^T(i,:)D(:,i) = 2}$$

The i^{th} row of D^T has only two non-zero elements -1 & 1

$$D^T(i,:)D(:,j)$$

Common vertex between e_1 & e_2 is v_2

$$D^T(1,:)D(:,2) = \underbrace{\begin{bmatrix} -1 & \overset{v_2}{1} & 0 & 0 \end{bmatrix}}_{e_1} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}}_{e_2} \, v_2 = 1$$

$$D^T(2,:)D(:,3) = \underbrace{\begin{bmatrix} 0 & \overset{v_2}{1} & 0 & -1 \end{bmatrix}}_{e_2} \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{e_3} \, v_2 = -1$$

Common vertex between e_2 & e_3 is v_2

$$D^T(1,:)D(:,4) = \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}}_{e_1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{e_4} = 0$$

No common vertex between e_1 & e_4

$L_e(i,j) = D^T(i,:)D(:,j) \leftarrow$ The entries are $(i \neq j)$

$$= \begin{cases} +1, & \text{if edges } e_i \,\&\, e_j \text{ share a common vertex } v_K, \text{ \& both of them either enter or leave from } v_K. \\ -1, & \text{if edges } e_i \,\&\, e_j \text{ share a common vertex } v_K, \text{ \& one of them enters and other leaves from vertex } v_K \\ 0, & \text{if there is no common vertex between } e_i \,\&\, e_j. \end{cases}$$

- The diagonal entries of L_e are always 2.