ELL805

Lecture - 8

- **Some Results on Matrix Theory**

→ Let $A \in \mathbb{R}^{n \times n}$ be a matrix with eigenvalues $\lambda_1, \lambda_2 \cdots \lambda_n$.

- Determinant of A → $\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$
- Trace of A → $tr(A) = \sum_{i=1}^{n} a_{ii}$

$$= \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

→ If λ_i is an eigenvalue of A, then λ_i^2 is an eigenvalue of A^2.

$$Ax = \lambda_i x \qquad x : \text{eigenvector corresponding to } \lambda_i$$

$$A(Ax) = \lambda_i Ax$$

$$\Rightarrow A^2 x = \lambda_i^2 x$$

- Let A has 'k' distinct eigenvalues:

$$\lambda_1, \lambda_2 \cdots \lambda_k \qquad k < n$$

Characteristic polynomial $\alpha(s) = \det(sI - A)$

$$= (s-\lambda_1)^{\mu_1}(s-\lambda_2)^{\mu_2} \cdots (s-\lambda_k)^{\mu_k}$$

For $i = 1, 2 \cdots k$, $\quad \mu_i$: Algebraic Multiplicity of λ_i.

- For eigenvalue λ_i & eigenvector x_i relation:

$$A x_i = \lambda_i x_i \leftarrow$$

$$\boxed{\Rightarrow \ (A - \lambda_i I) x_i = 0}$$

The eigenvector x_i belongs to the null space of $(A - \lambda_i I)$.

$$x_i \in \mathcal{N}(A - \lambda_i I)$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad \lambda_i = 1 \qquad \alpha(s) = (s-1)^2$$

2 is Algebraic multiplicity (A.M) of 1.

$$(A - \lambda_i I) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$$x_i = \begin{bmatrix} * \\ 0 \end{bmatrix} \in \mathcal{N}(A - \lambda_i I)$$

The dimension of $\mathcal{N}(A - \lambda_i I)$ is 1.

(One can use Ran-Nullity Theorem)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then is only one distinct eigenvalue $\lambda_i = 1$

$$\alpha(s) = (s-1)^3$$

$$\Rightarrow \text{A.M. of } \lambda_i = 1 \text{ is } \underline{\underline{3}}.$$

$$(A - 1\cdot I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix}$$

The $\dim\left(\mathcal{N}(A - \lambda_i I)\right) = 2$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{ Two distinct eigenvalue}$$

$$\lambda_1 = 2 \qquad \lambda_2 = 1$$

$$\alpha(s) = (s-2)^2(s-1)$$

$$\begin{cases} \text{A.M. of } \lambda_1 = 2 \text{ is } 2 \rightarrow \text{G.M} = 1 \\ \text{A.M of } \lambda_2 = 1 \text{ is } 1 \rightarrow \text{G.M} = 1 \end{cases}$$

$$(A - \lambda_1 I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathcal{N}(A - \lambda_1 I) \rightarrow \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}$$

$$\dim \mathcal{N}\left((A - \lambda_1 I)\right) = 1.$$

For $\lambda_2 = 1$

$$(A - \lambda_2 I) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dim\left(\mathcal{N}\left(\left(A - \lambda_2 I\right)\right)\right) = 1 \rightarrow \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix}$$

- The $\mathcal{N}(A - \lambda_i I)$ is called '<u>eigen space</u>' of λ_i.

- The dimension of $\mathcal{N}(A - \lambda_i I)$ is called "<u>Geometric multiplicity</u>" of λ_i.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \lambda_1 = 1 \rightarrow \begin{matrix} A.M. = 3 \\ G.M = 2 \end{matrix}$$

- <u>Result</u>: A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if and only if for <u>every</u> eigenvalue λ_i of A, the A.M. & G.M. of λ_i coincide.

$$\boxed{\text{For every } \lambda_i \rightarrow A.M. = G.M.}$$

- What about an matrix $A \in \mathbb{R}^{n \times n}$ with 'n' distinct eigenvalues?

- Two vectors v_1 & v_2 are <u>orthogonal</u>

  if $\boxed{v_1^T v_2 = 0}$ $\quad v_i \in \mathbb{R}^n$

- <u>Orthonormal</u> vectors when $\|v_i\|_2 = 1$ &

  they satisfy $v_1^T v_2 = 0$

- Orthogonal Matrix :

  A matrix Q whose columns are <u>orthonormal</u>, then Q is called orthogonal matrix.

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$$

$$\left.\begin{aligned} [Q]_{ii} &= q_i^T q_i = \|q_i\|_2 = 1 \\ \text{For } i \neq j \rightarrow [Q]_{ij} &= q_i^T q_j = 0 \end{aligned}\right\}$$

$$\boxed{Q^T Q = I} \Rightarrow \boxed{Q^{-1} = Q^T}$$

- Diagonalization $\rightarrow$ $V^{-1} A V = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

  $V \rightarrow$ the eigenvector of A.

$$\left\{\begin{aligned} v^T A = \lambda_i v^T \quad &\leftarrow v^T \text{ is left eigenvector of } A \\ A v_i = \lambda_i v_i \quad &\leftarrow v_i \text{ is right eigenvector of } A \end{aligned}\right.$$

- Let $A \in \mathbb{R}^{n \times n}$ satisfies $A = A^T$ (symmetric)

- A has 'n' distinct eigenvalues.

- The left & right eigenvectors of A are orthogonal.

- For every symmetric matrix $A \in \mathbb{R}^{n \times n}$, there exists an orthogonal matrix Q s.t.

$$Q^T A Q = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

The columns of Q are the right eigenvectors of A.

- Define a matrix

rank-1 matrices

$$\rightarrow \quad J := \mathbb{1}\mathbb{1}^T \qquad \mathbb{1} := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$J_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow \text{rank}(J_2) = 1$$

$$J_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \text{rank}(J_3) = 1$$

$$J_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \rightarrow \text{rank}(J_4) = 1$$

- The number of non-zero eigenvalue of $J = \mathbb{1}\mathbb{1}^T$

$$tr(J) = \sum J_{ii}$$

$$= \underbrace{\lambda_1}_{= n} + \underbrace{\lambda_2}_{0} + \cdots + \underbrace{\lambda_n}_{0} = n$$

- Since J is rank-1 matrix its null space dimension would be $n-1$.

$\rightarrow$ The eigenvectors corresponding to '0' eigenvalues of J.

$\downarrow$

$$J_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow (J - 0 \cdot I) v_i = 0$$

$$\Rightarrow J v_i = 0$$

- Any vector which belongs to the null space of J, is the eigenvector corresponding to '0' eigenvalue.