ELL 805

Lecture - 9

$$J = \mathbb{1}\mathbb{1}^T \qquad \mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The eigenvalues of J

$$\operatorname{tr}(J) = \sum_{i=1}^{n} J_{ii} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

$$= n$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \underbrace{\underset{v_2}{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} \quad \underset{v_3}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}}_{\text{eigenvectors corresponding to 0 eigenvalues of } J}$$

i.e. $J v_2 = 0$

$J v_3 = 0$

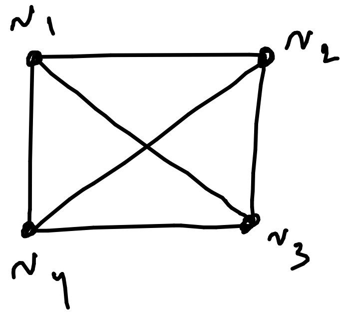
$$3I - J = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \rightarrow \underset{v_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$V = \begin{bmatrix} | & | & | \\ \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 \\ | & | & | \end{bmatrix} \qquad \tilde{v}_i = \frac{v_i}{\|v_i\|_2}$$

$$V^{-1} J V = \begin{bmatrix} n & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

n : is the non-zero eigenvalue of J.

→ Consider a complete graph :



$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Adjacency matrix K_4

$J_n = \mathbb{1}\mathbb{1}^T$

$$\boxed{A(K_n) = J_n - I_n}$$

K_n : complete graph on 'n' vertices.

Eigenvalues of $A(K_n)$ =

$V^{-1} A(K_n) V$ =

$$V^{-1}(J_n - I_n)V = V^{-1} J_n V - I$$

$$= \begin{bmatrix} n & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix} - \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} n-1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{bmatrix}$$

Since the eigenvalues of $A(K_n)$ & $V^{-1}A(K_n)V$ are equal, we have the following results.

→ **Result**

The eigenvalues of $A(K_n) = \begin{cases} n-1, & \text{with multiplicity } 1 \\ -1, & \text{with multiplicity } n-1. \end{cases}$

→ **Result**

The eigenvalues of $\alpha I + \beta J$ are $= \begin{cases} \alpha, & \text{with multiplicity } n-1 \\ \alpha + n\beta, & \text{with multiplicity } 1. \end{cases}$

α & β are scalars

→ **Result**

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of Laplacian matrix, corresponding to a graph G on 'n' vertices. Then, G is connected if and only if $\lambda_{n-1} > 0$.

Proof

First show that $\mathcal{N}(D^T) = \mathcal{N}(L)$

$\uparrow$ incidence matrix $\quad$ $\uparrow$ Laplacian matrix

Let us first show that $\mathcal{N}(L) \subseteq \mathcal{N}(D^T)$.

Let $\quad w \in \mathcal{N}(L)$

$$Lw = 0$$

$$\Rightarrow DD^Tw = 0$$

$$\Rightarrow \underbrace{w^T D}_{y^T} \underbrace{D^T w}_{y} = 0$$

$$\Rightarrow y^Ty = 0 \quad \Rightarrow y = 0 \quad \Rightarrow D^Tw = 0$$

$$\Rightarrow w \in \mathcal{N}(D^T).$$

Since w is chosen arbitrarily for $\mathcal{N}(L)$,

$$\Rightarrow \mathcal{N}(L) \subseteq \mathcal{N}(D^T)$$

Now we will show that $\mathcal{N}(D^T) \subseteq \mathcal{N}(L)$

Let w be a vector which belongs to $\mathcal{N}(D^T)$

$\Downarrow$

$$D^Tw = 0$$

$$\Rightarrow DD^Tw = 0$$

$$\Rightarrow DD^Tw = 0$$

$$\Rightarrow Lw = 0$$

$$\Rightarrow w \in \mathcal{N}(L)$$

Since w is arbitrary

$$\mathcal{N}(D^T) \subseteq \mathcal{N}(L).$$

$\Rightarrow \mathcal{N}(D^T) = \mathcal{N}(L)$

The $\dim(\mathcal{N}(D^T)) = 1$ iff G is connected.

$\mathbb{1} \in \mathcal{N}(D^T)$

$\Rightarrow \dim(\mathcal{N}(L)) = 1$ iff G is connected.

(Row sum of L is zero) $\longrightarrow$ $\boxed{L \cdot \mathbb{1} = 0}$

$\Rightarrow$ The vector $\mathbb{1}$ belongs to the $\mathcal{N}(L)$ & hence, it is an eigenvector of L corresponding to '0' eigenvalue.

$\downarrow$

The G.M. of $\lambda_n = 0$ is 1.

$\downarrow$

A.M. of $\lambda_n = 0$ is 1. (Since L is symmetric)

$\Downarrow$

L is always diagonalizable

$\Downarrow$

A.M. of λ_i = G.M. of λ_i for all i.

- Result

Let G be a connected graph on 'n' vertices. Let the eigenvalues of L are arranged as follows $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$. Then, the eigenvalues of $L + \alpha J$ are $(J = \mathbb{1}\mathbb{1}^T)$

$$\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n = n\alpha.$$

Proof

Since L is symmetric, $\exists$ an orthogonal matrix P s.t.

$$P^T L P = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & O & \\ & & \ddots & & \\ & O & & \lambda_{n-1} & \\ & & & & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} | & | & & | & | \\ p_1 & p_2 & \cdots & p_{n-1} & p_n \\ | & | & & | & | \end{bmatrix} \rightarrow \begin{cases} p_i^T p_j = 0 \text{ for } i \neq 0 \\ p_i^T p_i = 1 \end{cases}$$

$$p_n = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \rightarrow \frac{1}{\sqrt{n}} \mathbb{1}.$$

consider a matrix $\left[\begin{array}{c:c} & \sqrt{n} \\ 0 & \sqrt{n} \\ & \vdots \\ & \sqrt{n} \end{array}\right]$

$$\left[\begin{array}{c:c} & \sqrt{n} \\ 0 & \sqrt{n} \\ & \vdots \\ & \sqrt{n} \end{array}\right] \underbrace{\begin{bmatrix} - & p_1^T & - \\ - & p_2^T & - \\ & \vdots & \\ - & p_n^T & - \end{bmatrix}}_{P^T} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & & \\ 1 & - & - & 1 \end{bmatrix}}_{J}$$

$$p_n^T = \left[\frac{1}{\sqrt{n}} \quad \frac{1}{\sqrt{n}} \quad \cdots \quad \frac{1}{\sqrt{n}}\right]$$

$$JP = \left[\begin{array}{c:c} & \sqrt{n} \\ 0 & \sqrt{n} \\ & \vdots \\ & \sqrt{n} \end{array}\right]$$

$$P^TJP = \begin{bmatrix} - & p_1^T & - \\ - & p_2^T & - \\ & \vdots & \\ - & p_n^T & - \end{bmatrix} \left[\begin{array}{c:c} & \sqrt{n} \\ 0 & \sqrt{n} \\ & \vdots \\ & \underset{\uparrow\ \sqrt{n}\cdot\mathbb{1}}{\sqrt{n}} \end{array}\right]$$

$$= \left[\begin{array}{c:c} & 0 \\ & 0 \\ 0 & 0 \\ & \vdots \\ & 0 \\ & n \end{array}\right] \qquad p_i^T(\sqrt{n}\cdot\mathbb{1}) = 0 \quad \text{for } i = 1, 2, \cdots n-1.$$

$$P^T(L + \alpha J)P = P^TLP + \alpha P^TJP$$

$$= \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{n-1} & \\ & & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & n\alpha \end{bmatrix}$$

Since the eigenvalues of

$$L + \alpha J \quad \& \quad P^T(L + \alpha J)P \quad \text{are}$$

equal, the result holds. ∎