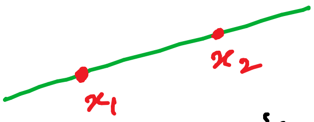
Part - 1

CONVEX SET & CONVEX FUNCTION

→ Lines & Line segments:

Let x_1 & x_2 be two points in $\mathbb{R}^n$.

Then, the points of the form:



$$y = \theta x_1 + (1-\theta)x_2 \quad \text{for } \theta \in \mathbb{R}$$

is a "line" passing through the points x_1 & x_2.

$$\left.\begin{array}{ll} \text{For } \theta = 0, & y = x_2 \\ \theta = 1, & y = x_1 \end{array}\right\}$$

- Line Segment: $y = \theta x_1 + (1-\theta)x_2$ with $\theta \in [0, 1]$.

→ Affine sets:

- A set S_a is affine if the line passing through any two points in S_a lies in S_a, i.e,

$$\boxed{y = \theta x_1 + (1-\theta)x_2 \in S_a \quad \text{for all } x_1 \in S_a,\ x_2 \in S_a \ \& \ \theta \in \mathbb{R}.}$$

- Affine combination of points:

For a set of points: $x_1, x_2 \cdots, x_k$, the points of the form:

$$\boxed{y = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \quad \text{with} \quad \sum_{i=1}^{k} \theta_i = 1}$$

is an affine combination of points x_i.

- If S_a is an affine set, and $x_i \in S_a$, then the points of the form:

$$y = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \quad \text{with} \quad \sum_{i=1}^{k} \theta_i = 1$$

also belong to S_a.

- If S_a is an affine set and $x_0 \in S_a$, then the set

$$\mathcal{V} := S_a - x_0 := \{ x - x_0 \mid x \in S_a \} \quad \cdots \; (*)$$

is a "subspace" (closed under vector addition & scalar multiplication).

$\downarrow$

To see this, choose two points $\begin{cases} v_1 \in \mathcal{V} \\ v_2 \in \mathcal{V} \end{cases}$

with scalars $\alpha, \beta \in \mathbb{R}$.

From the definition of $\mathcal{V}$ in $(*)$, the point v_1 & v_2 can be expressed as follows:

$$\left.\begin{aligned} v_1 &= x_1 - x_0 \\ v_2 &= x_2 - x_0 \end{aligned}\right\} \text{ with } x_1 \in S_a \;\&\; x_2 \in S_a$$

$$x_1 = v_1 + x_0$$
$$x_2 = v_2 + x_0$$

Consider a point : $\alpha_1 v_1 + \alpha_2 v_2 + x_0$

$$= \alpha_1 (x_1 - x_0) + \alpha_2 (x_2 - x_0) + x_0$$

$$= \alpha_1 x_1 + \alpha_2 x_2 - \alpha_1 x_0 - \alpha_2 x_0 + x_0$$

$$= \alpha_1 x_1 + \alpha_2 x_2 + (1 - \alpha_1 - \alpha_2) x_0$$

Since $x_0, x_1 \,\&\, x_2 \in S_a$ and $\alpha_1 + \alpha_2 + 1 - \alpha_1 - \alpha_2 = 1$,

the point $\alpha_1 v_1 + \alpha_2 v_2 + x_0 \in S_a$

Then, it follows from the definition ⊛ that the point $\alpha_1 v_1 + \alpha_2 v_2 \in \mathcal{V}$.

- From the definition ⊛, it also follows that an affine set S_a can be expressed as:

$$\boxed{S_a = \mathcal{V} + x_0 := \{ v_i + x_0 : v_i \in \mathcal{V} \}}$$

A subset + an offset x_0

Note that the choice of vector space $\mathcal{V}$ does not depend on x_0. x_0 could be any point in S_a.

- <u>Dimension of an affine set</u>

  The dimension of an affine set S_a is equal to the dimension of subspace $\mathcal{V}$ when

  $$\mathcal{V} = S_a - x_0 \text{ with } x_0 \in S_a .$$

- <u>Affine Hull</u> :

  Consider a set $S \subseteq \mathbb{R}^n$. Then the set of all affine combination of points of S is called the $\underbrace{\text{"affine hull"}}_{\text{denoted as } aff(S)}$ of S, i.e.

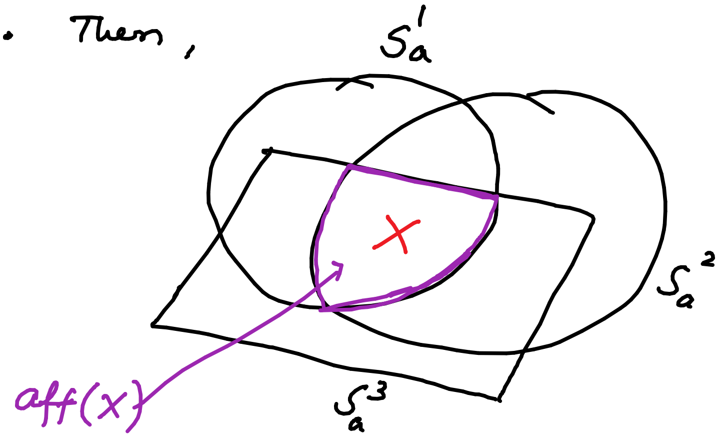
$$\boxed{aff(S) := \left\{ \sum_{i=1}^{k} \theta_i x_i \;:\; x_i \in S \text{ and } \sum_{i=1}^{k} \theta_i = 1 \right\}}$$

- <u>$aff(S)$</u> is also a set, and it is <u>the smallest affine set</u> containing S.

  $\downarrow$

  Assume that there are 'n' affine sets S_a^i containing a set X. Then,

  $$aff(X) = \bigcap_{i=1}^{n} S_i^a$$



- An affine set $S_a \subseteq \mathbb{R}^n$ is a set of the form $S_a = \mathcal{V} + x_0$ where $\mathcal{V}$ is a subspace.

  Note that the subspace $\mathcal{V}$ is uniquely determined by the affine set S_a i.e. the elements of $\mathcal{V}$ are of the form:

  $$v_i = x_i - x_0, \text{ where } x_i \in S_a \text{ and } x_0 \in S_a.$$

  $\downarrow$

  Here, $\mathcal{V}$ is called the <u>subspace parallel to</u> S_a.

- <u>Dimension of $\mathrm{aff}(X)$:</u>

  Since $\mathrm{aff}(X)$ is itself an affine set containing $X \subseteq \mathbb{R}^n$, we can associate a subspace $\mathcal{V}$ parallel to $\mathrm{aff}(X)$, as discussed above. Then, the dimension of $\mathrm{aff}(X)$ i.e.

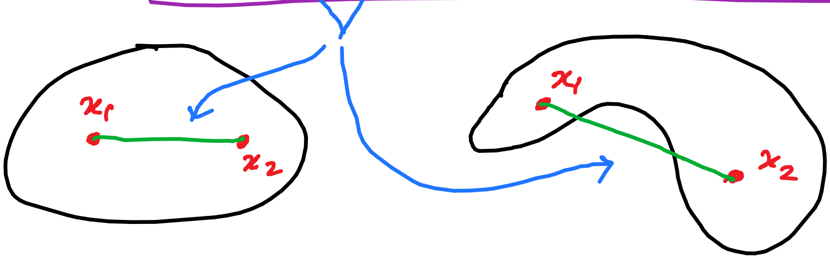
  $$\boxed{\dim(\mathrm{aff}(X)) := \dim(\mathcal{V})}$$

  $\uparrow$ dimension of subspace $\mathcal{V}$,

→ <u>Convex Sets</u> :

A set S_c is called convex if the <u>line segment</u> between any two points in S_c lies in S_c, i.e. for any $x_1 \in S_c$ & $x_2 \in S_c$

$$\boxed{\theta x + (1-\theta)x_2 \in S_c \quad \text{with} \quad \theta \in [0,1]}$$



Convex Set

Non-convex set

- Convex Combination of points:

A point y of the form:

$$\boxed{\begin{gathered} y = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \quad \text{with} \\ \theta_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{k} \theta_i = 1 \end{gathered}}$$

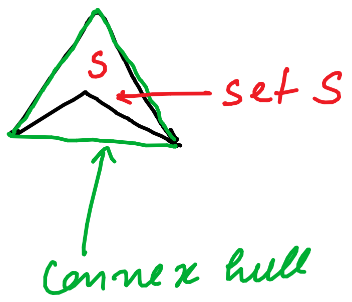
is called a <u>convex combination</u> of the points x_i, $i = 1, 2 \cdots k$.

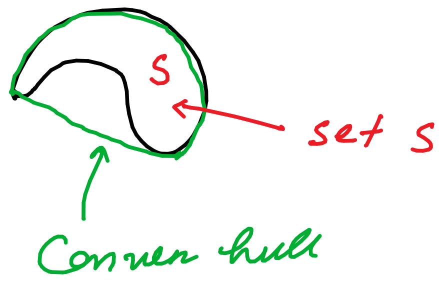
- **Convex Hull :**

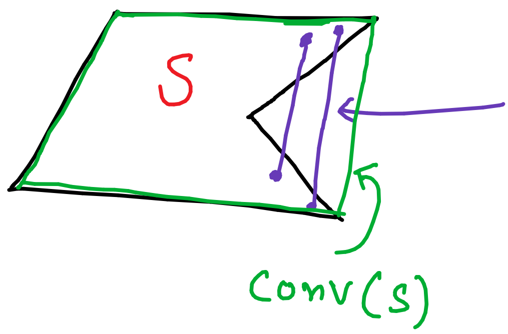
Convex hull of a set S, denoted as $conv(S)$, is the set of all convex combinations of points in S, i.e :

$$\boxed{conv(S) := \left\{ y = \sum_{i=1}^{n} \theta_i x_i \;\middle|\; x_i \in S,\ \theta_i \geq 0,\ \sum_{i=1}^{n} \theta_i = 1 \right.}$$

- The convex hull of S ($conv(S)$) itself is a convex set. It is the smallest possible convex set that contains the set S.







According to the defn, the points on these lines are part of convex hull of S.

→ Convexity Preserving Operations:

1. The intersection of any collection of convex sets is convex.

2. The vector sum of two convex sets S_1 & S_2 is convex, i.e. $S_1 + S_2$ is convex.
$\hookrightarrow \{x_1 + x_2 \mid x_1 \in S_1 \text{ \& } x_2 \in S_2\}$

3. The set αS is convex for any convex set S & scalar α. Further, if α_1 & α_2 are positive scalars & S is convex set, then

$$(\alpha_1 + \alpha_2) S = \alpha_1 S + \alpha_2 S$$

4. The closure & interior of a convex set are convex.

5. The image & inverse image of a convex set under an affine function $(Ax+b)$ are convex.

→ The convex hull of a compact set is compact.

→ CONES:

A set K is said to be a "Cone" if for all $x \in K$ and $\theta \geq 0$, we have

$$\boxed{\theta x \in K}$$

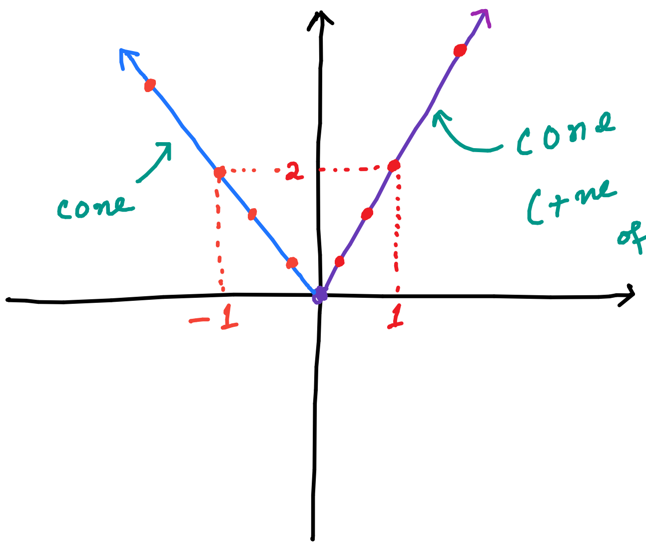
Let $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\theta = 0 \rightarrow x = 0$

$\theta = 0.1 \rightarrow x = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$

$\theta = 1 \rightarrow x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\theta = 10 \rightarrow x = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$



cone (the extension of a line)

rays

Let $x = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\theta = 0 \rightarrow x = 0$

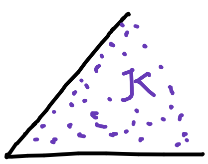
$\theta = 0.1 \rightarrow x = \begin{bmatrix} -0.1 \\ 0.2 \end{bmatrix}$

$\theta = 1 \rightarrow x = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

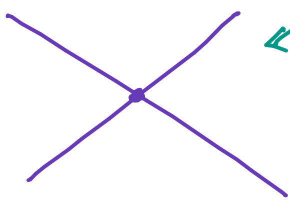
$\theta = 10 \rightarrow x = \begin{bmatrix} -10 \\ 20 \end{bmatrix}$

A ray $\{\theta x_i \text{ for } \theta \geq 0\}$ is a cone.

→ A cone K is a "convex cone" if K is convex & it is a cone.



Convex cone



Two lines passing through origin.

Non-convex cone

→ For a given set $S \subseteq \mathbb{R}^n$, the cone generated by S, denoted as cone(S), is the set of all non-negative combination of elements of S i.e.

$$\boxed{\text{cone}(S) := \left\{ y = \underbrace{\sum_{i=1}^{k} \theta_i x_i}_{\text{Conic combination of points } x_1, x_2 \ldots x_k} \;\middle|\; \theta_i \geq 0,\ x_i \in S \ \& \ k > 0 \right\}}$$

Boyd's book represents cone(S) as conic hull of a set S.

- Example:

Consider two points $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in plane. Let the set

$$S = \{x_1, x_2\} \subseteq \mathbb{R}^2$$

The cone generated by S is

$$\text{cone}(S) = \{ y = \theta_1 x_1 + \theta_2 x_2 \mid \theta_i \geqslant 0 \}$$

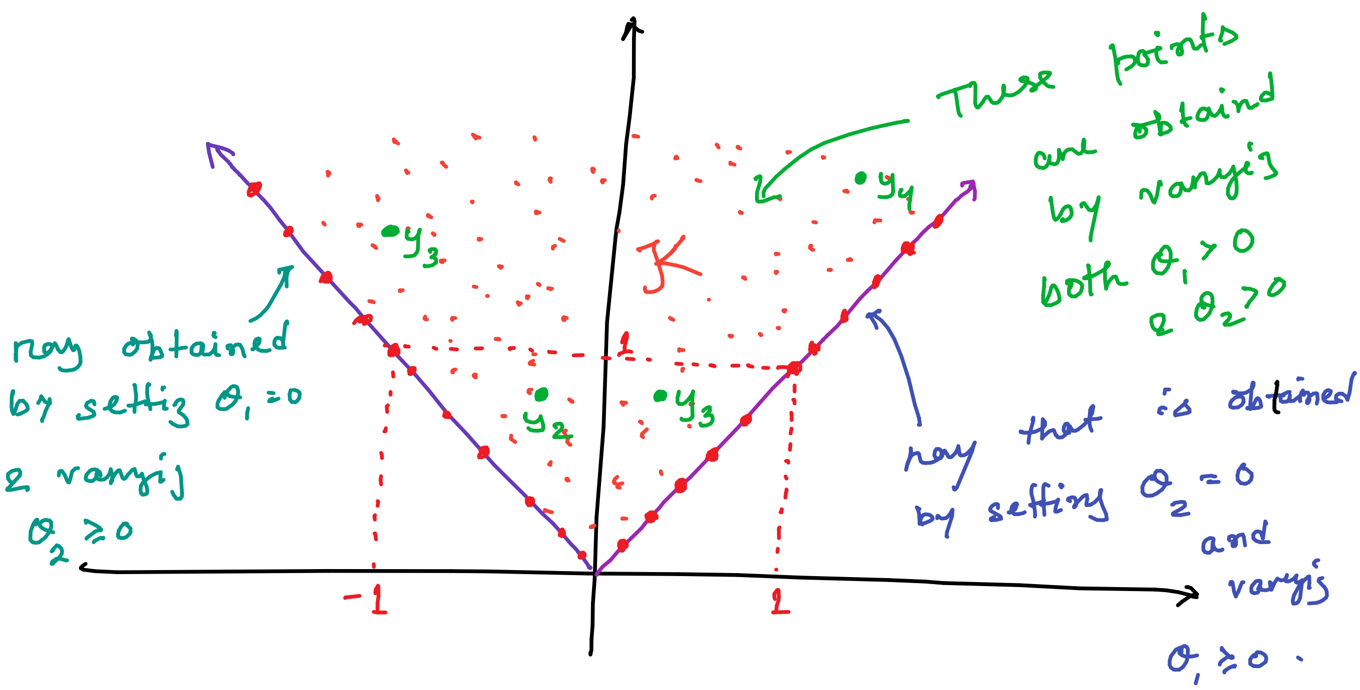
$$\theta_1 = 1 \qquad \theta_2 = 2$$

$$y_1 = \begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}$$

$$y_2 = 0.5\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-0.5\\1.5\end{bmatrix}$$

$$y_3 = \begin{bmatrix}1\\1\end{bmatrix} + 0.5\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}0.5\\1.5\end{bmatrix}$$

$$y_4 = 5\begin{bmatrix}1\\1\end{bmatrix} + 0.5\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}4.5\\5.5\end{bmatrix}$$



→ Cone Preserving Operations :

1. The intersection of a collection of cones is a cone.

2. The vector sum of two cones K_1 & K_2 is cone, i.e.

$$K_1 + K := \{ x_1 + x \mid x_1 \in K_1, \; x_2 \in K_2 \}$$ is a cone.

3. The closure of a cone $(cl(K))$ is a cone.

4. The image & inverse image of a cone under linear transformation is a cone.