VECTOR SPACE & MATRIX THEORY

Part-1

- vector $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

  $\mathbb{R}$: set of real numbers

  n : No of elements in x

  x_i : i^{th} component of x

- Matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$

- Inner product of two vectors $x \in \mathbb{R}^n$ & $y \in \mathbb{R}^n$ is : $\boxed{x^T y = \sum_{i=1}^{n} x_i y_i}$

  x^T : Transpose of x

- Two vectors x & y are said to be "Orthogonal" if $\boxed{x^T y = 0}$.

- The notation: $x > 0$ is $x_i > 0$

  $x - y > 0$ is $x_i - y_i > 0$

  $x \geq 0$ is $x_i \geq 0$

- A set S can be written as :

  $$S := \{ x \mid x \text{ satisfies } R \}$$

  i.e. collection of all x which satisfy property R.

- If S is a set & x is its element then we write $x \in S$.
- The union of two sets S_1 & S_2 is $S_1 \cup S_2$
- The intersection of two sets S_1 & S_2 is $S_1 \cap S_2$.
- A set V is said to be a "vector space" over a field $\mathbb{R}$ if it satisfies:

$$\alpha_1 v_1 + \alpha_2 v_2 \in V$$

for every $v_i \in V$ & $\alpha_i \in \mathbb{R}$

- A non-empty subset S of $\mathbb{R}^n$ $(S \subset \mathbb{R}^n)$ is called "subspace" if

$$\alpha_1 x_1 + \alpha_2 x_2 \in S$$

for every $x_i \in S$ & $\alpha_i \in \mathbb{R}$.

- Affine Set :

An affine set in $\mathbb{R}^n$ is a translated subspace i.e. the set X is affine if it can be represented as

$$X ::= \{ \bar{x} + x \mid x \in S \}$$

$$|||$$

$$X = \bar{x} + S$$

where S is a subspace of $\mathbb{R}^n$ & $\bar{x} \in \mathbb{R}^n$.

$\downarrow$

S is also called the "subspace parallel to X".

- <u>Linearly independent vectors</u>

  A set of vectors $\{x_1, x_2 \cdots x_n\}$ with $x_i \in \mathbb{R}^n$ are said to be linearly independent if there exist no scalars $\alpha_1, \alpha_2 \cdots \alpha_n$, with atleast one of which is non-zero, such that $\sum_{i=1}^{n} \alpha_i x_i = 0$.

- <u>Basis of a vector space:</u>

  A "basis" of a vector space $V \subseteq \mathbb{R}^n$ a set of vectors : $\{v_1, v_2 \cdots v_m\}$ s.t.

  (i) $v_1, v_2 \cdots v_m$ are linearly independent, and

  (ii) every vector $v \in V$ can be represented as : $v = \sum_{i=1}^{m} \alpha_i v_i$,

  where α_i is a scalar.

- The number of elements in a basis of a vector space V is called <u>dimension of V</u>.

- The space $\{0\}$ is said to be of "zero - dimension".

- **Dimension of Affine Set**

  The dimension of an affine set $X = \bar{x} + S$ is the dimension of the associated subspace S.

- **Orthogonal Complement of a Subspace :**

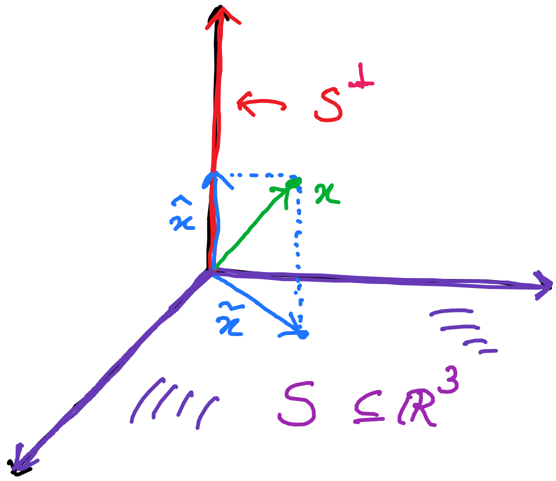
  Given a subspace S, the set of vectors that are orthogonal to all the elements of S is a subspace, denoted by $S^{\perp}$. i.e.

$$\boxed{S^{\perp} := \{ y \mid y^T x = 0 , \ \forall \ x \in S \}}$$

The subspace $S^{\perp}$ is called Orthogonal complement of S.

- If the subspace $S \subset \mathbb{R}^n$, then any vector $x \in \mathbb{R}^n$ can uniquely be decomposed as sum of a vector from S & a vector from $S^{\perp}$. i.e

$$\boxed{x = \tilde{x} + \hat{x} \quad \text{where } \tilde{x} \in S \ \& \ \hat{x} \in S^{\perp}.}$$



Any vector in S are of the form

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ 0 \end{bmatrix}$$

So $$\hat{x} = \begin{bmatrix} 0 \\ 0 \\ \hat{x}_3 \end{bmatrix} \in S^{\perp}$$

Then $$x = \tilde{x} + \hat{x}$$

- The decomposition of $x = \tilde{x} + \hat{x}$ where $\tilde{x} \in S$ & $\hat{x} \in S^{\perp}$ is called "orthogonal decomposition" of x w.r.t. subspaces : S & $S^{\perp}$.

  $\tilde{x}$: Orthogonal projection of x onto S

  $\hat{x}$: Orthogonal projection of x onto $S^{\perp}$.

The operator P that maps x to $\tilde{x}$

i.e. $P : x \to \tilde{x}$

is called "Orthogonal Projector" onto S.

- Addition of two vector spaces:

$$S_1 + S_2 = \{ x_1 + x_2 \mid x_1 \in S_1 \ \& \ x_2 \in S_2 \}$$

- For a subspace $S \subset \mathbb{R}^n$, we have

$$\boxed{S + S^{\perp} = \mathbb{R}^n} \quad \&$$

$$\boxed{\dim(S) + \dim(S^{\perp}) = n}$$

$\dim(\cdot)$: dimension

- Further, we have: $\boxed{(S^{\perp})^{\perp} = S}$.

- Subspaces associated with a matrix:

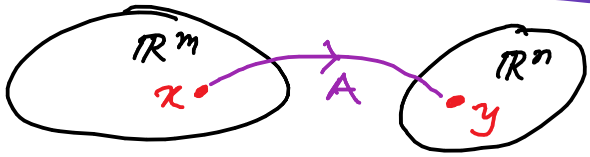
Let $A \in \mathbb{R}^{n \times m}$ matrix:

- Range Space of A:

$$\boxed{\mathcal{R}(A) := \{ y \in \mathbb{R}^n \mid y = Ax, \ x \in \mathbb{R}^m \}}$$

- Null Space of A:

$$\boxed{\mathcal{N}(A) := \{ x \in \mathbb{R}^m \mid Ax = 0 \}}$$



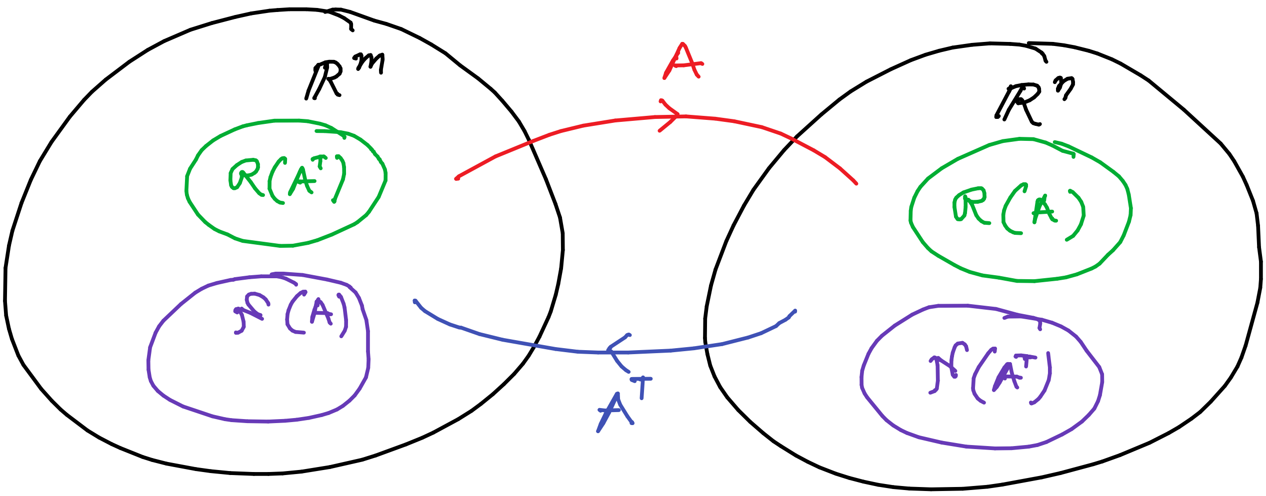
- $$\text{rank}(A) = \dim(\mathcal{R}(A))$$
$$|||$$
maximal number of linearly independent rows/columns

- $A \in \mathbb{R}^{n \times m}$ is said to be of "Full Rank"

  if $$\boxed{\text{rank}(A) = \min(m, n)}$$
  $$|||$$
  All rows or colums of A are linearly independent.

- The two subspaces $\mathcal{R}(A)$ & $\mathcal{N}(A)$ associated with A satisfies following relation:

$$\boxed{\mathcal{R}(A) = \left[\mathcal{N}(A^T)\right]^{\perp}} \qquad A \in \mathbb{R}^{n \times m}$$

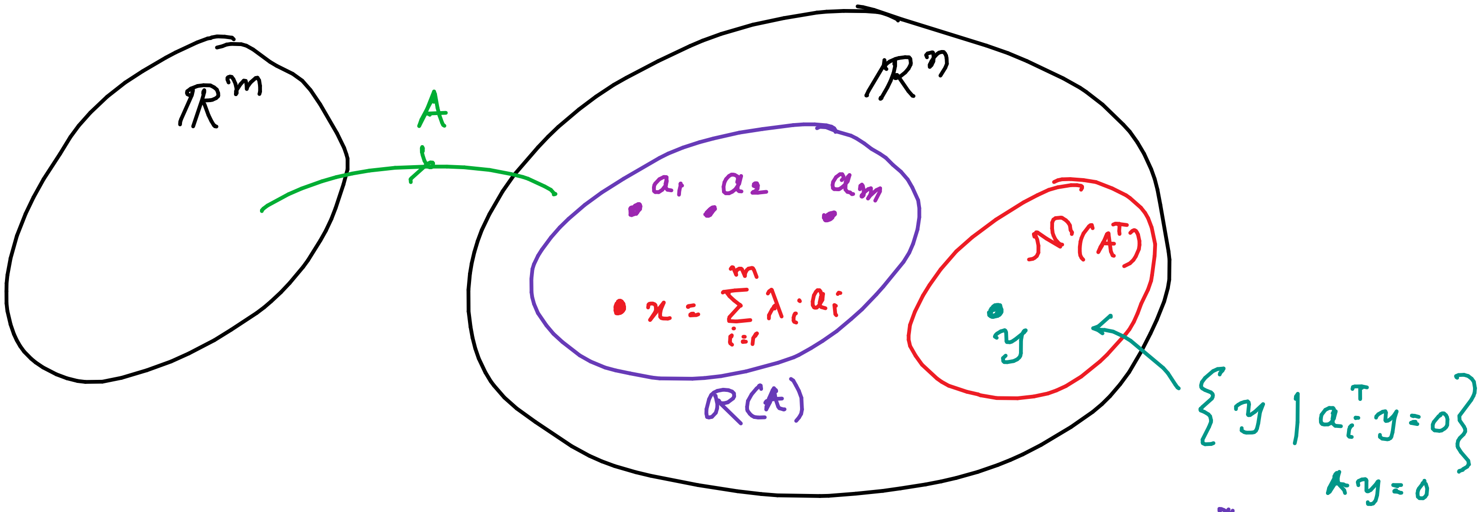


- Another Interpretation of $\mathcal{R}(A) = [\mathcal{N}(A^T)]^\perp$.

For the given vectors $a_1, a_2 \cdots a_m$ with $a_i \in \mathbb{R}^n$ (the columns of A) & a vector $x \in \mathbb{R}^n$, we have:

$$\left\{ \begin{array}{l} x^T y = 0 \text{ for all } y \text{ such that } a_i^T y = 0, \forall i \\ \Updownarrow \text{ if \& only if} \\ \text{there exist some scalars } \lambda_1, \lambda_2 \cdots \lambda_m \\ \text{such that } x = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m \end{array} \right.$$

A special case of Farkas' Lemma



$$\underbrace{\begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}_{m \times n}}_{A^T} y = 0$$

- Further, $\boxed{\mathcal{N}(A) = [\mathcal{R}(A^T)]^{\perp}}$

- Rank-Nullity Theorem:

$$\boxed{\begin{array}{l} \dim[\mathcal{R}(A)] + \dim[\mathcal{N}(A)] = m \\ \text{for all } n \times m \text{ matrices} \end{array}}$$

- $\text{rank}(A^TA) = \text{rank}(A) = \text{rank}(AA^T)$

- Affine function

  - A function $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is said to be "affine" if it has the following form:

$$\boxed{\begin{array}{l} f(x) = a^T x + b \\ \text{for some } a \in \mathbb{R}^m \text{ \& } b \in \mathbb{R} \end{array}}$$

  - A function $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is said to affine if it has the following form:

$$\boxed{\begin{array}{l} f(x) = Ax + b \\ \text{for some } A \in \mathbb{R}^{n \times m} \text{ \& } b \in \mathbb{R}^n \end{array}}$$