MATRIX THEORY

→ Matrix Norms

A matrix norm $\|\cdot\|$ is a function that assigns a real non-negative number to each matrix $A \in \mathbb{R}^{n\times n}$, i.e.

$$\|\cdot\| : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$$

$$A \longmapsto \|A\|$$

satisfying the following properties:

(i) $\|A\| \geq 0$ & $\|A\| = 0$ when $A = 0$

(ii) $\|\alpha A\| = |\alpha| \, \|A\|$

(iii) $\|A+B\| \leq \|A\| + \|B\|$

(iv) $\|AB\| \leq \|A\| \cdot \|B\|$

Examples :

1. Frobenious norm of A

$$\|A\|_F := \left[\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right]^{1/2}$$

2. Induced matrix norm:

$$\|A\|_2 := \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

(Any other vector norm can be used)

Also known as "spectral norm"

3. $\|A\|_1 := \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right)$

(addition of absolute values of entries in a column & then taking the maximum)

4. $\|A\|_\infty := \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}| \right)$

(addition of $|a_{ij}|$ in a row, & then taking the maximum.

→ A vector norm & its induced matrix norm satisfy the following inequality:

$$\|Ax\| \le \|A\| \|x\|$$

if 2-norm is used to define induced norm, then

$$\|Ax\|_2 \le \|A\|_2 \|x\|_2$$

→ All vector & matrix norms are equivalent, i.e. for any two norms $\|\cdot\|$ & $\|\cdot\|'$, there exists a scalar γ such that

$$\|\cdot\| \le \gamma \|\cdot\|'$$

⇓

If a set in $\mathbb{R}^n$ is open/closed/bounded/compact w.r.t some norm, then it is open/closed/bounded/compact w.r.t. all other norms.

→ Following statements are equivalent:

(i) A matrix $A \in \mathbb{R}^{n \times n}$ is non-singular.

(ii) A matrix A^T is non-singular.

(iii) For every non-zero $x \in \mathbb{R}^n$, we have

$$Ax \neq 0.$$

(iv) For every $y \in \mathbb{R}^n$, there exists unique $x \in \mathbb{R}^n$ s.t. $y = Ax$

(V) The rows & columns of A are linearly independent.

→ A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called "symmetric positive definite" matrix $(A \succ 0)$ if

$$\boxed{x^T A x > 0, \forall x \in \mathbb{R}^n \ \& \ x \neq 0,}$$

& called "symmetric positive semi-definite" $(A \succeq 0)$

$$\boxed{x^T A x \geq 0, \forall x \in \mathbb{R}^n.}$$

→ Some Results

1. A matrix A is symmetric positive definite if and only if it is invertible & its inverse is symmetric positive definite matrix.

2. Let $A \succcurlyeq 0$ & $B \succcurlyeq 0$ (symmetric positive semidefinite).

Then $\boxed{(A+B) \succcurlyeq 0}$. Further if

if one of them, say $B \succ 0$, then

$\boxed{(A+B) \succ 0}$ (positive definite)

3. Let $A \in \mathbb{R}^{n \times n}$ be: $A \succcurlyeq 0$ & $T \in \mathbb{R}^{m \times n}$

be any matrix. Then:

$$\boxed{TAT^T \succcurlyeq 0}.$$

Further, if $A \succ 0$, & T is invertible

then $\boxed{TAT^T \succ 0}$.

3. Let $A \in \mathbb{R}^{n \times n}$ be : $A \succ 0$. Then, there

exists positive scalars $\bar{\gamma}$ & $\underline{\gamma}$ s.t.

$$\boxed{\underline{\gamma}\, \|x\|^2 \leq x^T A x \leq \bar{\gamma}\, \|x\|^2, \quad \forall\, x \in \mathbb{R}^n}$$

4. A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive semi-definite if and only if all of its eigenvalues are non-negative, & it is positive definite if and only if all of its eigenvalues are positive.

- The spectral radius of A is:

$$\rho(A) = \max_i |\lambda_i| \qquad \lambda_i : i^{th} \text{ eigenvalue of } A.$$

- Since the roots of a polynomial depends continuously on its coefficients, the eigenvalues of a square matrix A depend continuously on the elements of A. Hence, $\rho(A)$ is a continuous function of A.

→ Results on convergence of Linear iterative Methods:

1. For any induced matrix norm $\|\cdot\|$ for a matrix A, we have:

$$\boxed{\lim_{k\to\infty} \|A^k\|^{1/k} = \rho(A) \leq \|A\|}$$

Further, for any given $\epsilon > 0$, there exists an induced matrix norm $\|\cdot\|$ s.t.

$$\boxed{\|A\| = \rho(A) + \epsilon}$$

2. For a square matrix A, we have

$$\boxed{\lim_{k\to\infty} A^k = 0 \quad \text{if and only if} \quad \rho(A) < 1.}$$

- For a symmetric matrix A, there exists an orthogonal matrix Q & a diagonal matrix Λ such that

$$Q^T A Q = \Lambda \quad (\text{or } A = Q \Lambda Q^T)$$

The columns of Q are eigenvectors & the diagonal elements of Λ are eigenvalues of A.

Spectral decomposition of A.

→ Singular Value Decomposition (SVD)

For a given non-zero matrix $A \in \mathbb{R}^{n \times m}$ with rank $(A) = r$, following decomposition always holds:

$$A = U \Sigma V^T$$

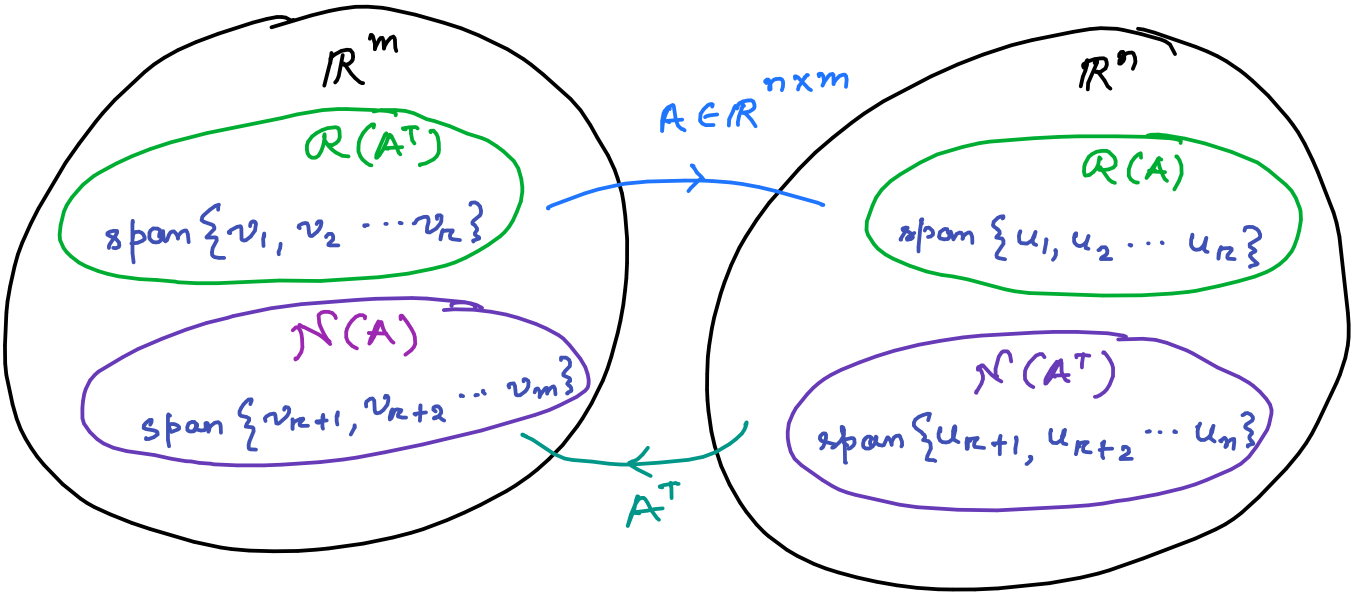
where $U \in \mathbb{R}^{n \times n}$, $V \in \mathbb{R}^{m \times m}$ are orthogonal & $\Sigma \in \mathbb{R}^{n \times m}$ is a diagonal matrix with non-negative diagonals:

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix} \quad \text{with } \sigma_i > 0$$

$$\begin{cases} U^T U = U U^T = I \\ V V^T = V^T V = I \end{cases}$$

σ_i : Singular value of A.

- The SVD of A characterize all 4 fundamental subspaces associated with A :



→ Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$ be the singular values of $A \in \mathbb{R}^{n \times m}$. Then $\boxed{\|A\|_2 = \sigma_1}$.

Furth $\boxed{\|A\|_2 = \|A^T\|_2 = \sigma_1}$

→ Let $A \in \mathbb{R}^{n \times m}$ with rank$(A) = r$ has following SVD : $A = U \Sigma V^T$ where

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}$$

Construct U_r by taking 1^{st} 'r' columns of U

V_r by taking 1^{st} 'r' columns of V

$$\Sigma_r = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}$$

The the Pseudoinverse of A (denoted as A^+) is:

$$\boxed{A^+ = V_r \Sigma_r^{-1} U_r^T}$$

→ Let $A \in \mathbb{R}^{n \times m}$ with $n \geq m$ & $\text{rank}(A) = m$.

Then $$\boxed{A^+ = (A^T A)^{-1} A^T}$$

→ Let $A \in \mathbb{R}^{n \times m}$ with $n \leq m$ & $\text{rank}(A) = n$.

Then $$\boxed{A^+ = A^T (A^T A)^{-1}}$$

→ $A^T A$ & AA^T are symmetric matrices, & they have the same non-zero eigenvalues (including multiplicity). Further,

$$\boxed{A^T A \succcurlyeq 0 \quad \& \quad AA^T \succcurlyeq 0}$$

→ Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ be the non-zero singular values of matrix A. Then, the non-zero eigenvalues of $A^T A$ & AA^T are:

$$\boxed{\sigma_1^2, \ \sigma_2^2 \ \ldots \ \sigma_r^2}$$

- Hence :

$$\boxed{\|A\|_2 = \sigma_1 = \sqrt{\lambda_{max}(A^TA)} = \sqrt{\lambda_{max}(AA^T)}}$$

largest singular value of A.

- $$\boxed{\begin{aligned}\|A\|_F &= \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}|a_{ij}|^2} \\ &= \sqrt{tr(A^TA)} = \sqrt{tr(AA^T)} \\ &= \sqrt{\sum_{i=1}^{r}\sigma_i^2}\end{aligned}}$$

where r is rank(A) & σ_i : singular values of A

$tr(A)$: trace of A = sum of diagonal entries of A.