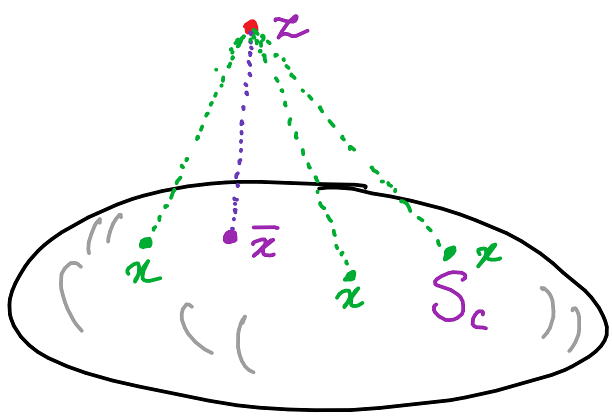
PROJECTION THEOREM

- **Projection on a convex set**

Let $z \in \mathbb{R}^n$ be fixed vector. Consider the problem of finding a vector $\bar{x} \in S_c$, where S_c is a closed convex set, such that the distance between $\bar{x}$ & z is minimum i.e.

$$
P1 \begin{cases} \min & f(x) = \|z - x\|_2^2 \\ \text{s.t.} & x \in S_c \end{cases}
$$

Then the vector $\bar{x}$, which is solution of P_1 is called the projection of z on S_c



- Results (Projection Theorem)

Let S_c be a nonempty & closed convex subset of $\mathbb{R}^n$. Then following statements hold.

(i) For every $z \in \mathbb{R}^n$, there exists a unique $\bar{x} \in S_c$ that minimizes $f(x) = \|z - x\|_2^2$ over all $x \in S_c$. The vector $\bar{x}$ is called projection of z on S_c, & denoted as $\text{Proj}(z)$ i.e. $\bar{x} = \text{Proj}(z)$.

(ii) Given some $z \in \mathbb{R}^n$, a vector $\bar{x} \in S_c$ is equal to the projection $\text{Proj}(z)$ if & only if

$$\boxed{(z - \bar{x})^T (x - \bar{x}) \leq 0 \quad , \forall x \in S_c}$$

(iii) The mapping function $\text{Proj} : \mathbb{R}^n \to S_c$ is continuous & nonexpansive, i.e.

$$\boxed{\|\text{Proj}(z_1) - \text{Proj}(z_2)\| \leq \|z_1 - z_2\| \quad , \forall z_1, z_2 \in \mathbb{R}^n}$$

(iv) In the case when S is a subspace, a vector $\bar{x} \in S$ is equal to $\text{Proj}(z)$ if & only if $z - \bar{x}$ is orthogonal to S

ie

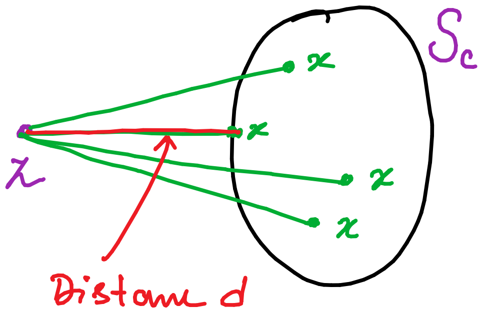
$$\boxed{(z - \bar{x})^T x = 0 \quad \forall x \in S}$$

(v) The distance function $d : \mathbb{R}^n \to \mathbb{R}$, defined as

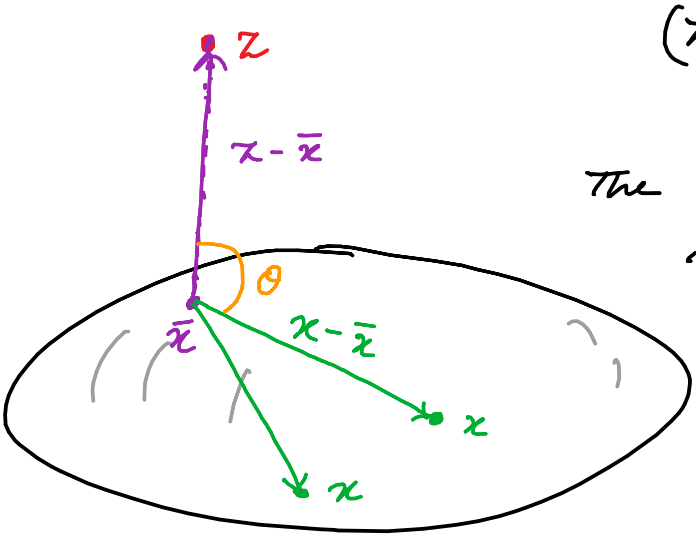
$$\boxed{d(z, S_c) := \min_{x \in S_c} \|x - z\|_2^2}$$

distance between pt. z and the convex set S_c

is a convex function.

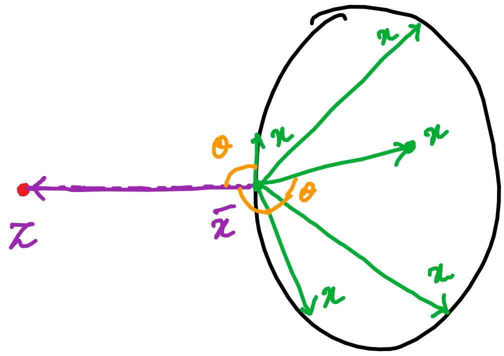


• Geometrical interpretation of part - ii

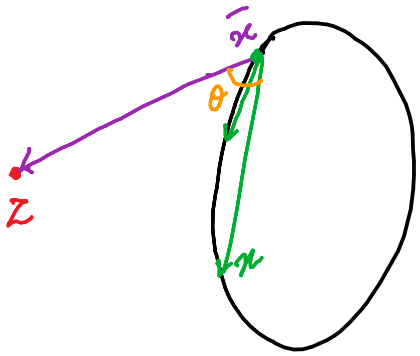


$(z - \bar{x})(x - \bar{x}) \leq 0$

The angle θ between two vectors $(z - \bar{x})$ & $(x - \bar{x})$ is greater than $90°$ (obtuse angle.)



$\bar{x}$ is projection of z, since all the vectors $(x-\bar{x})$ are making obtuse angle θ with the vector $(z-\bar{x})$



$\bar{x}$ is not a projection of z, since the vectors $(x-\bar{x})$ are making acute angle $(\theta < 90^\circ)$ with the vector $(z-\bar{x})$, & hence

$$(z-\bar{x})^T(x-\bar{x}) > 0$$