REAL ANALYSIS

Part - 1

- ### Norm

A "norm" $\|\cdot\|$ on $\mathbb{R}^n$ is a function that maps a vector $x \in \mathbb{R}^n$ to $\|x\| \in \mathbb{R}$, i.e. $\|\cdot\| : \mathbb{R}^n \longrightarrow \mathbb{R}$, which satisfies following properties:

(i) $\|x\| \geq 0 \quad \forall\, x \in \mathbb{R}^n$,

(ii) $\|x\| = 0$ if & only if $x = 0$

(iii) $\|\alpha x\| = \alpha \|x\|$ for every $\alpha \in \mathbb{R}$ & $x \in \mathbb{R}^n$

Triangle inequality → (iv) $\|x+y\| \leq \|x\| + \|y\|$, $\forall\, x, y \in \mathbb{R}^n$

- Two norm of a vector

$$\|x\|_2 := \left[\sum_{i=1}^{n} |x_i|^2 \right]^{1/2} \quad \text{for} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

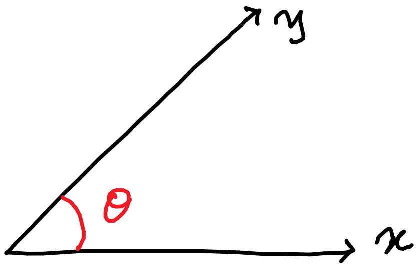
- ### Pythagorean Theorem

For any two orthogonal vectors $x \in \mathbb{R}^n$ & $y \in \mathbb{R}^n$ we have:

$$\boxed{\|x+y\|_2^2 = \|x\|_2^2 + \|y\|_2^2}$$

- For two non-zero vectors $x \in \mathbb{R}^n$ & $y \in \mathbb{R}^n$, the angle between them is:

$$\boxed{\theta = \cos^{-1}\left(\frac{x^T y}{\|x\|_2 \|y\|_2}\right)}$$



- x & y are said to be orthogonal when $\theta = 90^\circ$

$$|||$$

$$\boxed{x^T y = 0}$$

- Assuming that $\|x\|_2 = 1$ & $\|y\|_2 = 1$, the angle between x & y are said to be:

(i) "acute angle" if $\underbrace{0^\circ < \theta < 90^\circ}$

$\downarrow$

$0 < \cos\theta < 1$

$$|||$$

$$\boxed{x^T y > 0}$$

(ii) "Obtuse angle" if $\underbrace{90^\circ < \theta < 180^\circ}$

$-1 < \cos\theta < 0$

$$|||$$

$$\boxed{x^T y < 0}$$

- ## Cauchy-Schwarz Inequality

For any two vectors $x \in \mathbb{R}^n$ & $y \in \mathbb{R}^n$

we have : $\boxed{|x^T y| \leq \|x\|_2 \|y\|_2}$

- Some others vector norms:

i) ∞ - norm : $\|x\|_\infty := \max_i |x_i|$

ii) 1 - norm or ℓ_1 - norm : $\|x\|_1 := \sum_{i=1}^{n} |x_i|$

→ ## Sequences :

- A "sequence" in $\mathbb{R}$ is a series of real numbers, & is represented as:

$$\underbrace{\{x_k \mid k = 1, 2 \cdots\}}_{\text{Notation : } \{x_k\}} \quad \text{where } x_k \in \mathbb{R}.$$

Series of scalars

- Similarly, a "sequence" in $\mathbb{R}^n$ is a series of vectors :

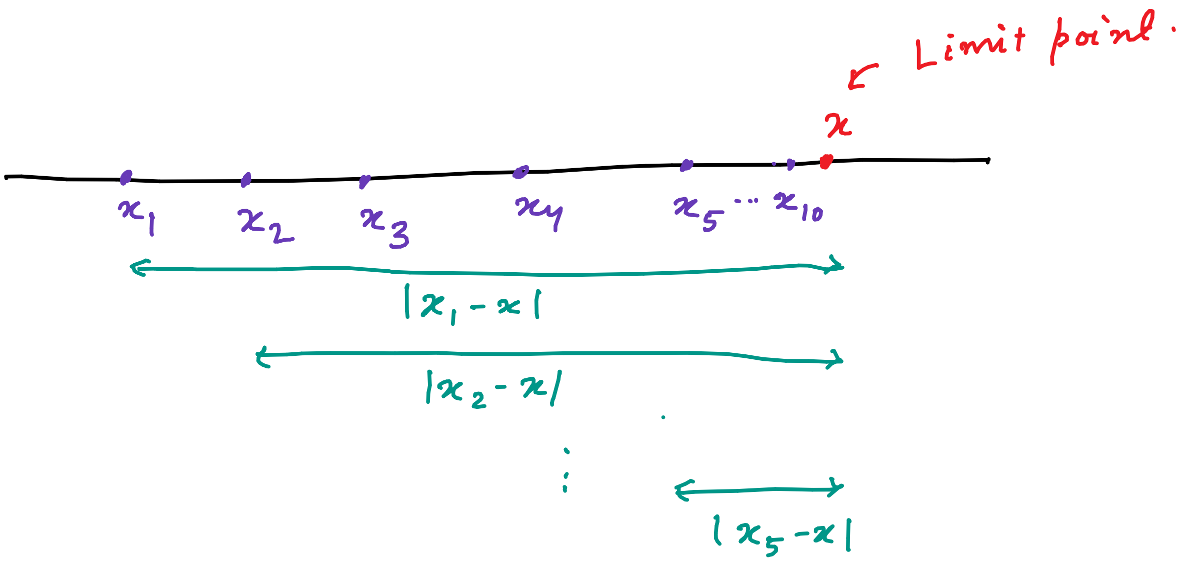
$$\{x_k \mid k = 1, 2 \cdots\} \quad \text{when } x_k \in \mathbb{R}^n$$

- **Convergence of a Sequence:**

A sequence $\{x_k\}$ of scalars is said to be "convergent" if there exists a scalar x s.t for every given $\epsilon > 0$:

$$\boxed{|x_k - x| < \epsilon \quad \text{for every } k > N}$$

where N is some integer & it depends on ϵ.



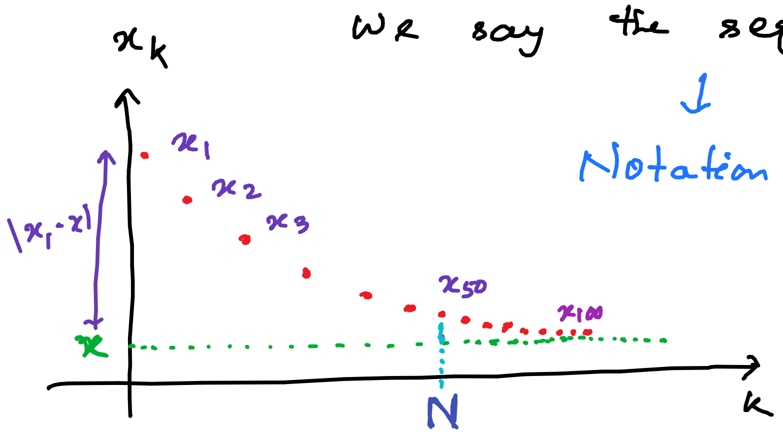
- The point x is called "limit point" of the sequence $\{x_k\}$.

$\downarrow$

We say the sequence $\{x_k\}$ converges to x.

$\downarrow$

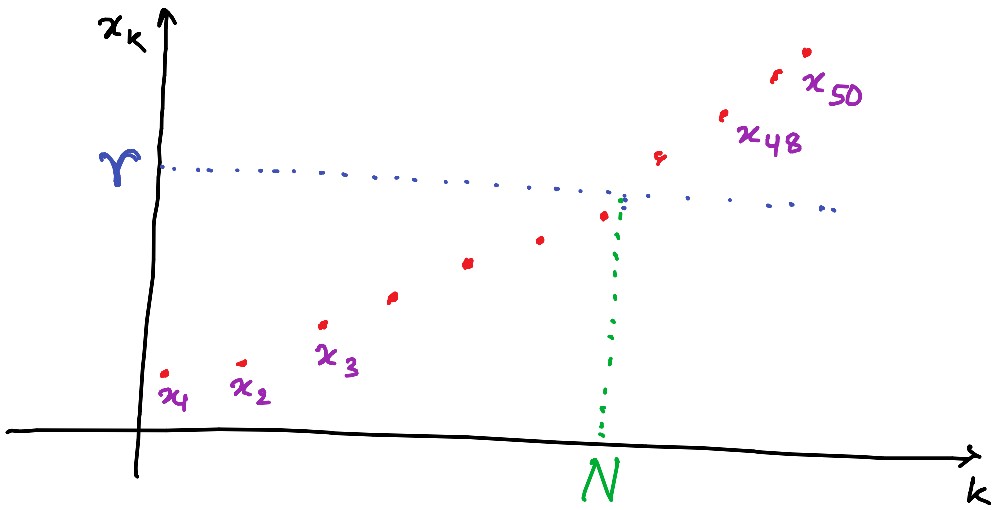
Notation: $\{x_k\} \to x$ or $\lim_{k \to \infty} x_k = x$



- Divergence Sequence

A sequence $\{x_k\}$ of real numbers is said to be "divergent" if for every given scalar r, there exists some N (depends on r) s.t.

$$\boxed{x_k \geq r \quad \text{for all} \quad k \geq N}$$



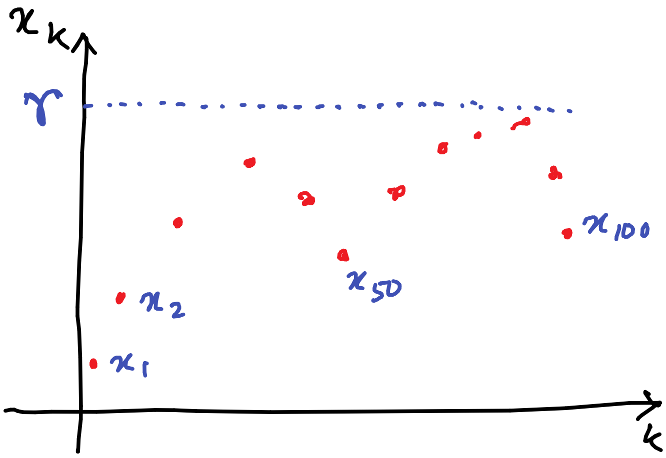
In this case, we say that the sequence $\{x_k\}$ is divergent & write it as $\{x_k\} \to \infty$

or $\lim_{k \to \infty} x_k = \infty$

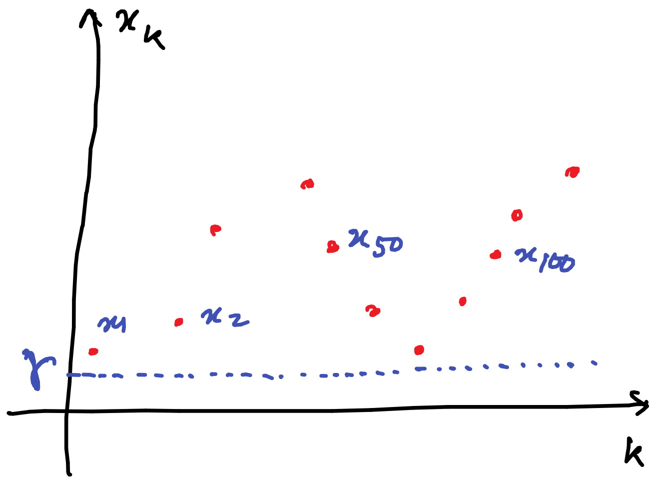
- A sequence is said to be "bounded above" if there exists some scalar γ s.t. $x_k \leq \gamma$ for all k.

$\downarrow$

Similarly $\{x_k\}$ is "bounded below" if $x_k \geq \gamma$ for all k.



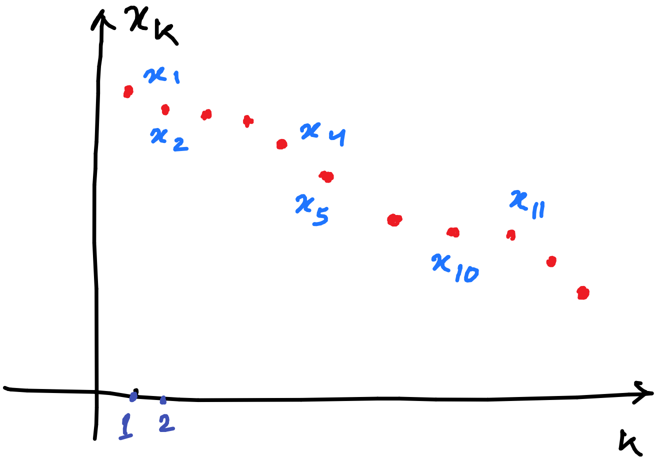
$\{x_k\}$ bounded above

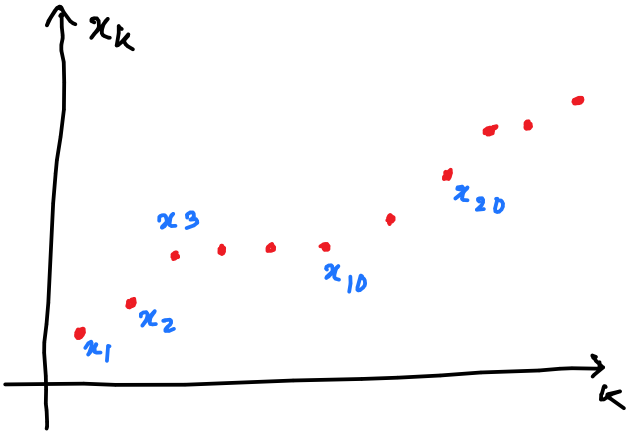


$\{x_k\}$ bounded below

- A sequence $\{x_k\}$ of scalars is said to be "monotonically non-increasing" sequence if $x_{k+1} \leq x_k$ for all k.

Similarly, $\{x_k\}$ is "monotonically nondecreasing" sequence if $x_{k+1} \geq x_k$ for all k.





Monotonically Nonincreasing $\{x_k\}$

$x_{k+1} \leq x_k$

Monotonically nondecreasing $\{x_k\}$

$x_{k+1} \geq x_k$

$\downarrow$

Notation for a sequence $\{x_k\}$ converging to x i.e. $\{x_k\} \to x$ & $\{x_k\}$ is monotonically nonincreasing sequence:

$\{x_k\} \downarrow x$

$\rightarrow$ $\{x_k\} \uparrow x$ for $\{x_k\} \to x$ & $\{x_k\}$ is monotonically nondecreasing.

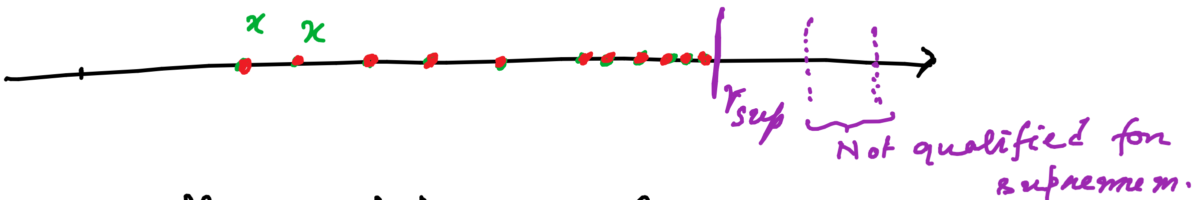
$\rightarrow$ **Result**

Every non-increasing or non-decreasing sequence $\{x_k\}$ converges, possibly to infinite. If $\{x_k\}$ is also bounded, then it converges to a finite real number.

- ## Supremum of a set

Consider a set S of scalars. Then, the "supremum" of S is the smallest scalar r_{sup} such that

$$\boxed{x \leq r_{sup}, \quad \forall\, x \in S} \quad \ldots\ *$$



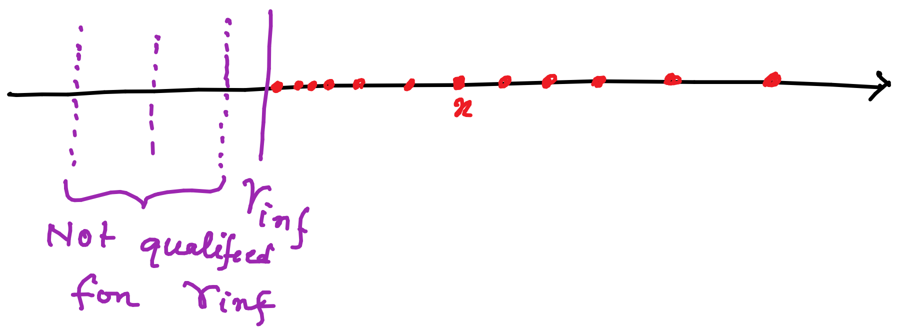
If no r_{sup} exists s.t. ⊛ holds, then we say that $\underbrace{\text{supremum of } S}_{\sup S}$ is ∞.

- ## Infimum of set

The "infimum" of a set S of scalars, is the largest scalar r_{inf} such that:

$$\boxed{x \geq r_{inf}, \quad \forall\, x \in S.}$$

If no such r_{inf} exists, then $\inf S = \infty$.

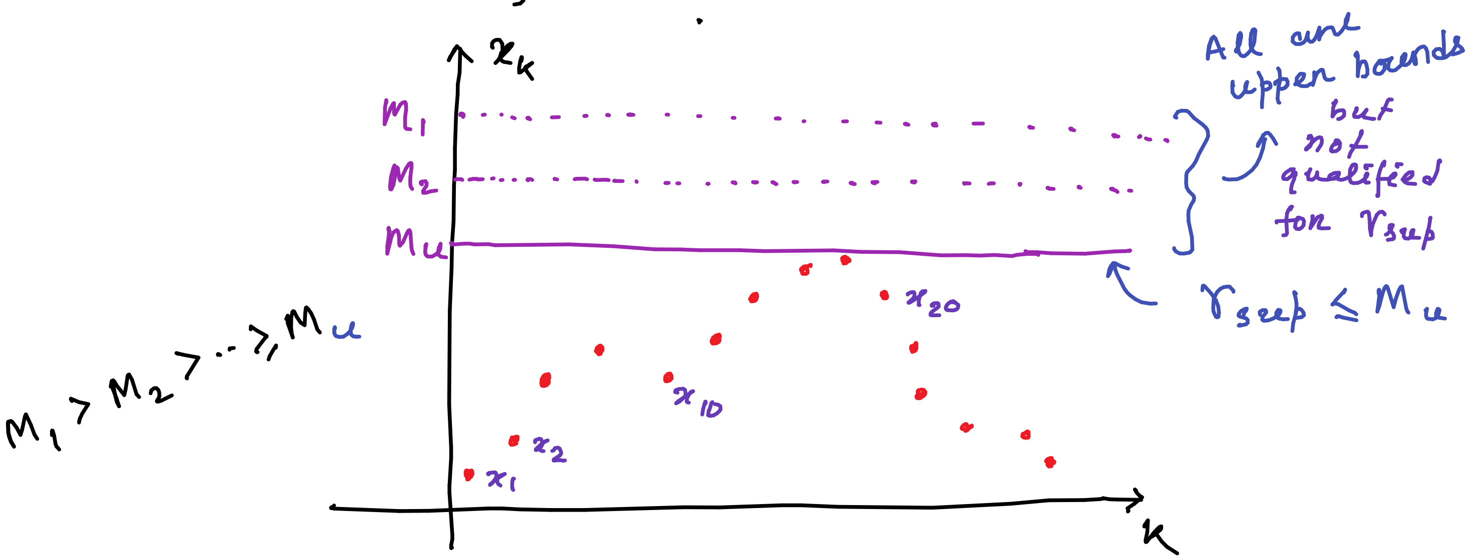


- **Supremum of a sequence:**

Let $\{x_k\}$ be a sequence in $\mathbb{R}$. Assume that $\{x_k\}$ is bounded above. Then, the "supremum" of $\{x_k\}$ is an upper bound γ_{sup} such that:

$$\gamma_{sup} \leq M_u$$

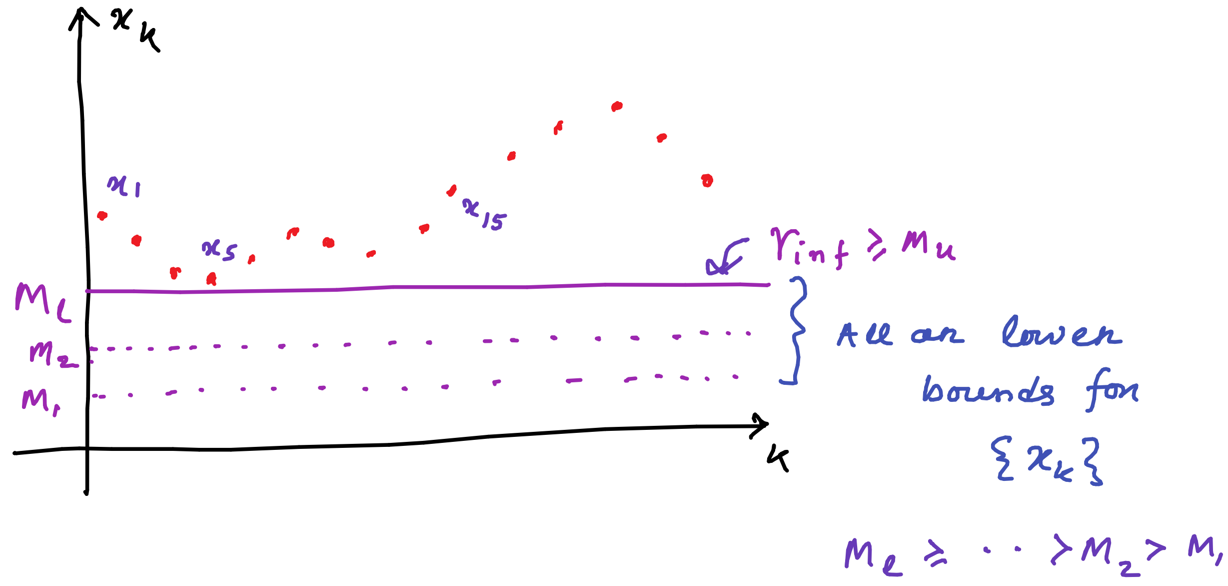
when M_u is the least upper bound of $\{x_k\}$.



- **Infimum of a Sequence:**

Consider a sequence $\{x_k\}$ of real numbers. Assume that $\{x_k\}$ is bounded below. Then, the "infimum" of $\{x_k\}$ is a lower bound γ_{inf} such that: $\gamma_{inf} \geq M_l$

where M_l is the greatest lower bound of $\{x_k\}$



- The infimum and supremum of $\{x_k\}$, if they exist, then they are unique.

Notations : $\inf\{x_k\} \leftarrow$ infimum of $\{x_k\}$

$\sup\{x_k\} \leftarrow$ supremum of $\{x_k\}$

→ For a given sequence $\{x_k\}$, define the following quantities :

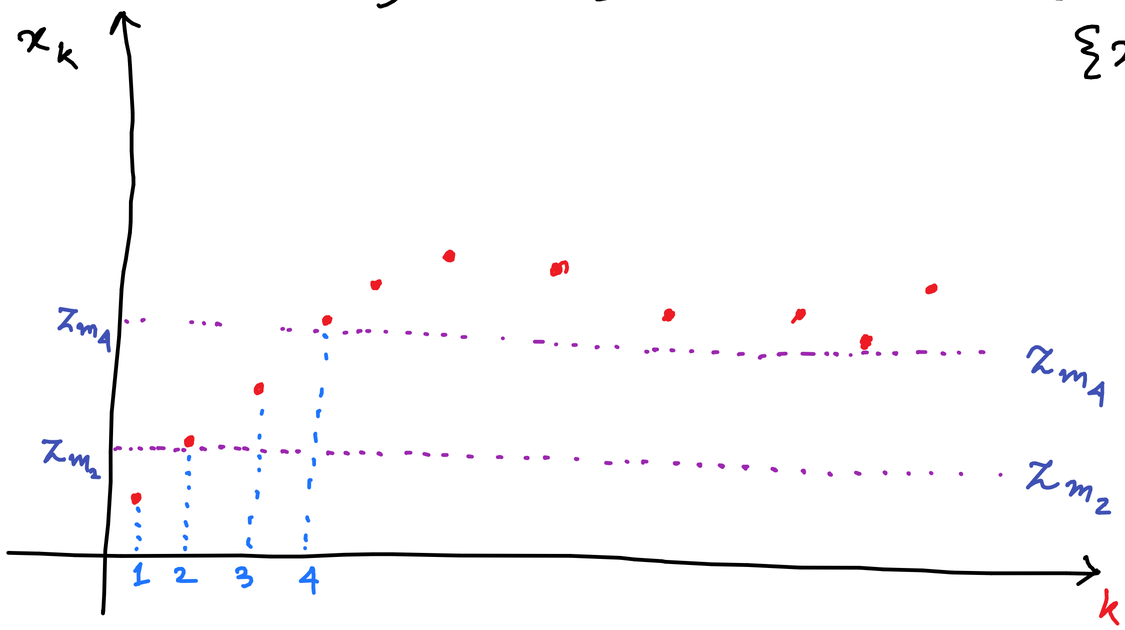
$$\boxed{\begin{aligned} y_m &:= \sup\{x_k \mid k \geq m\} \\ z_m &:= \inf\{x_k \mid k \geq m\} \end{aligned}}$$

For each given m, note that

y_m is z_m are some scalars.

Hence, we can form sequences:

$\{y_m\}$ & $\{z_m\}$ for every sequence $\{x_k\}$.



$z_{m_2} = \inf\{x_k \mid k \geq 2\}$

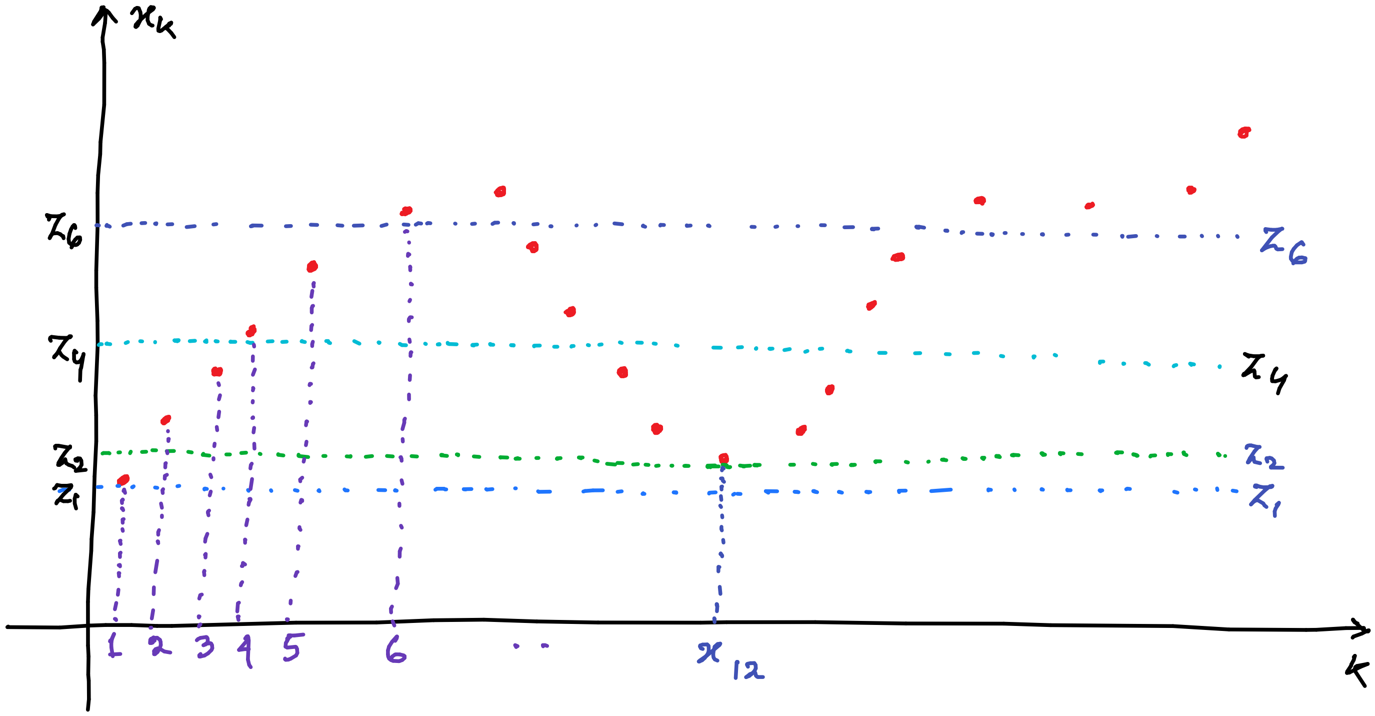
$z_{m_4} \quad \inf\{x_k \mid k \geq 4\}$

$\vdots$

$\rightarrow$ The sequence

$\{y_m\}$: Non-increasing sequence

$\{z_m\}$: Non-decreasing sequence



Since $z_m = \inf\{x_k \mid k \geq m\}$, it can be seen from the above fig. that

$$z_1 \leq z_2 \leq z_3 \leq z_4 \leq z_5 \leq z_6 \leq \cdots$$

$z_1 \downarrow \{x_k \mid k \geq 1\}$, $z_4 \downarrow \{x_k \mid k \geq 4\}$

Hence the sequence $\{z_m\}$ satisfies:

$$\boxed{z_{m+1} \geq z_m}$$

$\Downarrow$

$\{z_m\}$ is a non-decreasing sequence.

Similarly $\{y_m\}$ is a non-increasing sequence.

- Since the sequences $\{y_m\}$ & $\{z_m\}$ are non-increasing & non-decreasing, respectively, they have a limit point (possibly ∞).

This follows due to the previous result on non-increasing & non-decreasing sequences.

Notations (for given sequence $\{x_k\}$)

limit point of $\{y_m\}$ → $\limsup_{m\to\infty} x_m$

limit point of $\{z_m\}$ → $\liminf_{m\to\infty} x_m$

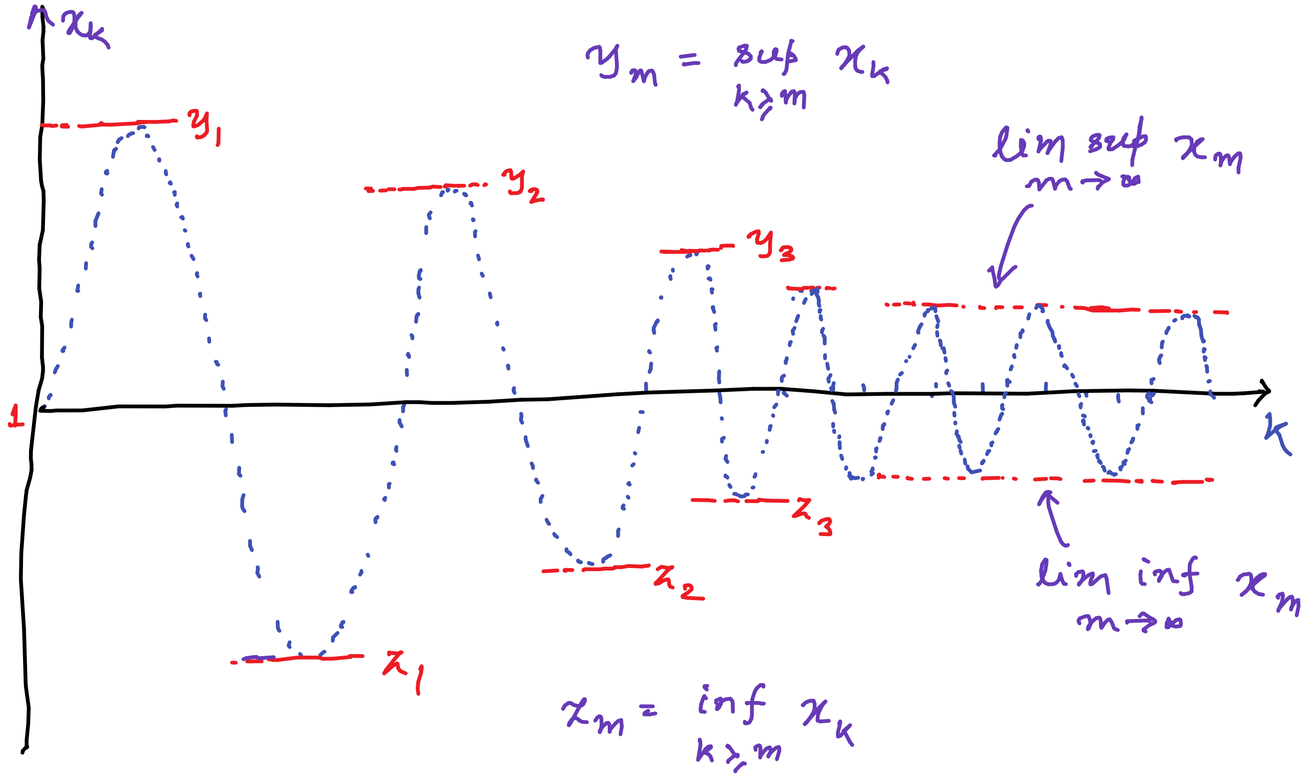
Defn

$$\limsup_{m\to\infty} x_m := \lim_{m\to\infty} \Big(\underbrace{\sup_{k\geq m} x_k}_{= y_m} \Big)$$

$$\liminf_{m\to\infty} x_m := \lim_{m\to\infty} \Big(\underbrace{\inf_{k\geq m} x_k}_{= z_m} \Big)$$

Called limit superior

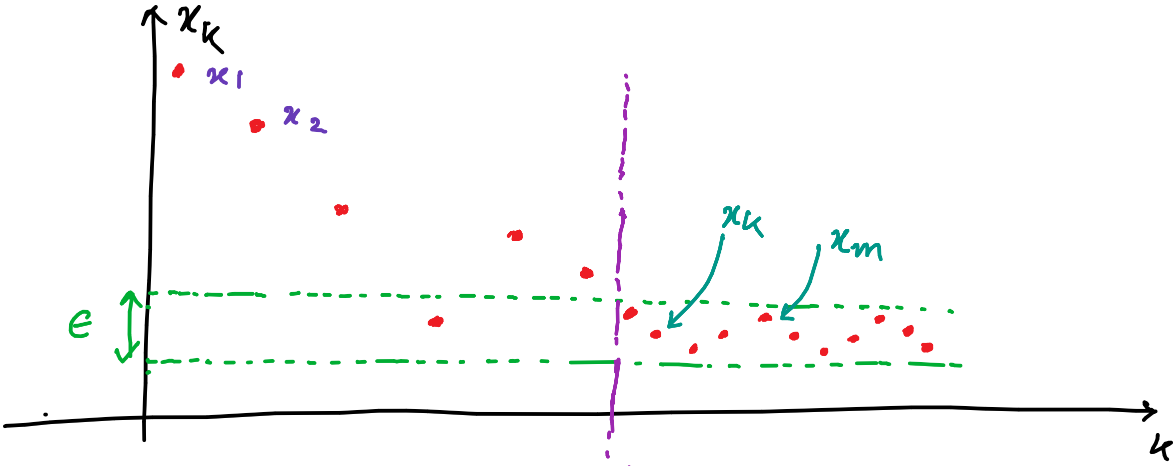
called limit inferior



→ Cauchy Sequence :

A sequence $\{x_k\}$ of scalars is called Cauchy sequence if for every given $\epsilon > 0$, there exists some N (depends on ϵ) such that

$$\boxed{|x_k - x_m| < \epsilon \quad \text{for all } k \geq N \ \& \ m \geq N}$$



$N \rightarrow$ After this point onwards

$$|x_k - x_m| < \epsilon \quad \forall\, k \geq N$$
$$\& \; m \geq N$$

→ Properties of Cauchy Sequence:

- Every Cauchy sequence is bounded.
- Every convergent sequence is Cauchy sequence.
- Every real Cauchy sequence is convergent.

→ Sequence of Vectors:

- A sequence of vectors $\{x_k\}$ in $\mathbb{R}^n$ i.e. $x_k \in \mathbb{R}^n$ for every k, is said to be convergent to a point $x \in \mathbb{R}^n$ if for every i, the ith component of $\{x_k\}$ converges to the ith component of x.

$$\{x_k\} = \begin{bmatrix} \{x_{k_1}\} \\ \{x_{k_2}\} \\ \vdots \\ \{x_{k_n}\} \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\{x_k\}$ $\quad$ x

$\{x_{k_i}\}$ is a sequence in $\mathbb{R}$

$\{x_{k_i}\} \to x_i$

$$\underbrace{\begin{bmatrix} 1+\frac{1}{k} \\ 2-\frac{1}{k} \end{bmatrix}}_{\{x_k\}} \xrightarrow{k\to\infty} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{x}$$

$x_k \to x$

on

$\lim_{k\to\infty} x_k = x$

- A sequence $\{x_k\}$ in $\mathbb{R}^n$ is bounded if each of its component sequence is bounded.

$$\{x_k\} = \begin{bmatrix} \{x_{k_1}\} \\ \{x_{k_2}\} \\ \vdots \\ \{x_{k_n}\} \end{bmatrix}$$

$\{x_{k_1}\}$ ← bounded sequence

$\{x_{k_n}\}$ ← bounded sequence

- The sequence $\{x_k\}$ in $\mathbb{R}^n$ is bounded if and only if there exists a scalar r such that

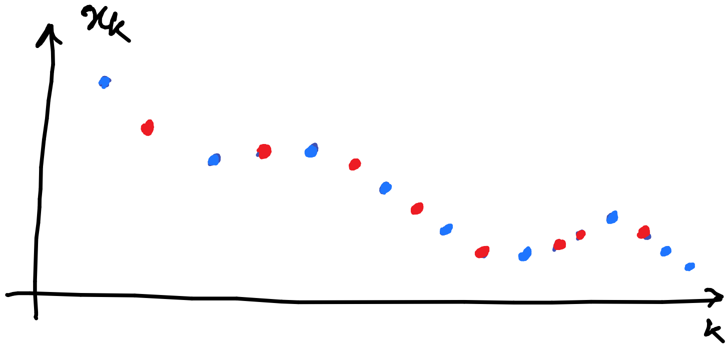
$$\boxed{\|x_k\| \leq r \quad \text{for all } k}$$

- Subsequence :

An infinite subset of a sequence $\{x_k\}$ is called subsequence of $\{x_k\}$.

$\Downarrow$

A subsequence itself can be viewed as a sequence.



$\leftarrow$ Sequence with blue dots. Its subsequence is only considering the red dots, which are also the elements of the sequence.

A sequence $\{x_k\} = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \dots\}$

$\downarrow$ Subsequence

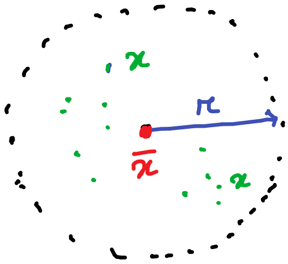
$\{1, \frac{1}{4}, \frac{1}{8}, \frac{1}{12} \dots\}$

$\nearrow$

The subsequence itself is a sequence.

→ For a given point $\bar{x} \in \mathbb{R}^n$ & $r \in \mathbb{R}_+$, we define the "open ball" as follows:

$$\mathcal{B}(\bar{x}, r) := \{ x \mid \|x - \bar{x}\| < r \}$$



- A sequence $\{x_k\}$ of vectors is "convergent" & has a "limit point": $\bar{x}$ i.e. $\lim_{k \to \infty} x_k = \bar{x}$ if for every given $\epsilon > 0$, there exists some N such that each $\boxed{x_k \in \mathcal{B}(\bar{x}, \epsilon)}$ i.e.

$$\boxed{\|x_k - \bar{x}\| < \epsilon \quad \text{for all } k > N}$$

Note that the points x_k which are inside the ball $\mathcal{B}(\bar{x}, \epsilon)$ is a subsequence of sequence $\{x_k\}$. Hence, the limit point of a sequence can also be defined as follows.

→ A vector $\bar{x} \in \mathbb{R}^n$ is said to be a "limit point" of a sequence $\{x_k\}$ in $\mathbb{R}^n$ if there exists a subsequence of $\{x_k\}$ (consider the subsequence inside the ball) that converges to the vector $\bar{x}$.

→ Some Results

- A bounded sequence of vectors in $\mathbb{R}^n$ is convergent if and only if it has a unique limit point.
- A sequence in $\mathbb{R}^n$ is convergent if and only if it is a Cauchy Sequence.
- Every bounded sequence in $\mathbb{R}^n$ has at least one limit point.

Bolzano-Weierstrass Theorem