REAL ANALYSIS

Part - 2

→ Topological Properties of Sets

- **Closure Point & Closure of a Set**

Consider a set $X \subseteq \mathbb{R}^n$. Then, the point x is a closure point of X if there exists a sequence $\{x_k\} \subset X$ that converges to the point x. The closure of X, denoted as $cl(X)$ is:

$$\boxed{cl(X) := \{x \mid x \text{ is a closure point of } X\}}$$

- **Closed Set** : A subset X of $\mathbb{R}^n$ is called closed if it is equal to its closure.

- **Open Set** : The set $X \subseteq \mathbb{R}^n$ is called open if its complement : $\{x \mid x \notin X\}$ is closed.

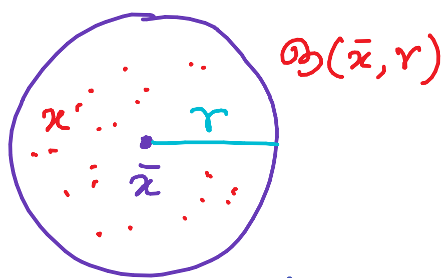
- <u>Bounded Set</u> : The set $X \subseteq \mathbb{R}^n$ is called bounded if there exists a scalar r such that $\|x\| \leq r$ for all $x \in X$.

- <u>Compact Set</u> : The set $X \subseteq \mathbb{R}^n$ is called compact if X is closed and bounded.

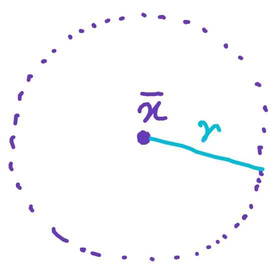
- For given $r > 0$ & $\bar{x} \in \mathbb{R}^n$, define the following sets in $\mathbb{R}^n$:

$$\mathcal{B}(\bar{x}, r) := \{ x \mid \|x - \bar{x}\| < r \}$$ ← Open set or Open ball

$$\mathcal{B}(\bar{x}, r) := \{ x \mid \|x - \bar{x}\| \leq r \}$$ ← Closed set or closed ball

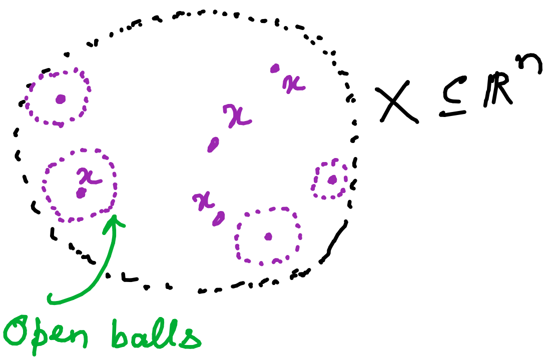


Closed ball



open ball

- A set $X \subseteq \mathbb{R}^n$ is <u>open</u> if and only if for every $x \in X$, there is an open ball with center at x, is contained in X.



- Neighborhood of a vector:

  The neighborhood of a vector x is an open set that contains x.

- Interior point of a set:

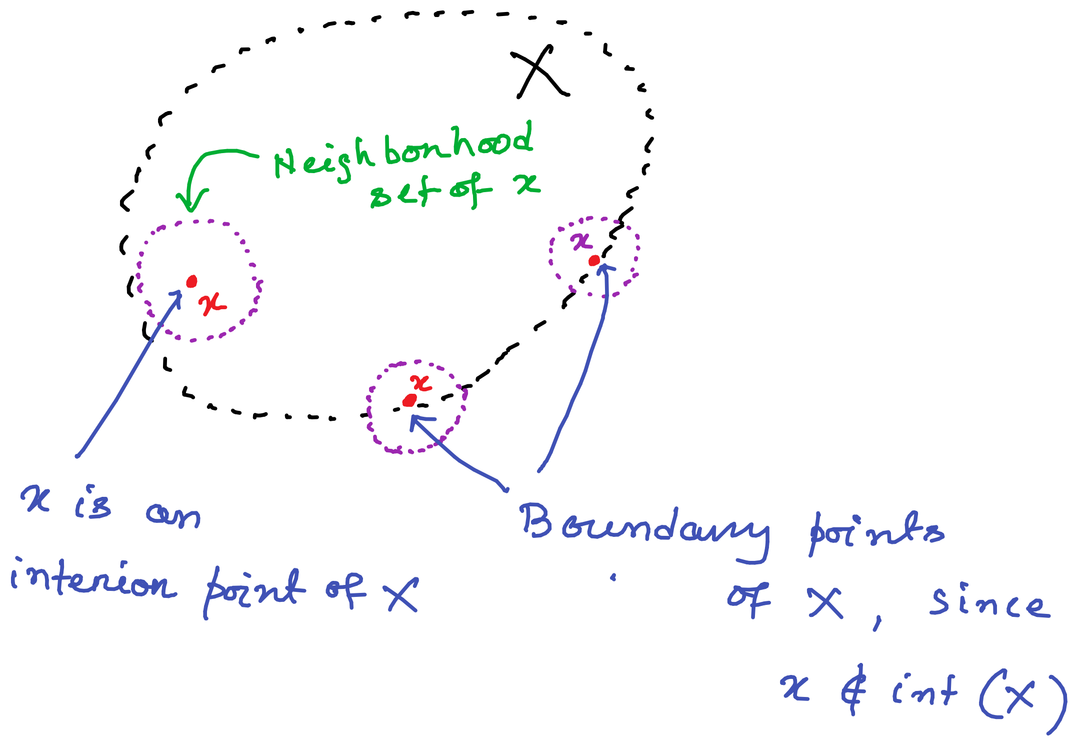
  Consider a set $X \subseteq \mathbb{R}^n$. Then, a point $x \in X$ is said to be an interior point of X if there exists a neighborhood of x that is entirely contained in X.

- The set of all interior points of X is called "interior of X", denoted as $int(X)$.

- Boundary Point:

  A vector $x \in cl(X)$, which is not an interior point of X is called boundary point of X.

- The set of all boundary points of X is called "boundary of X".



→ Some topological properties of Sets

Consider the sets $S_1, S_2, S_3 \cdots S_N$

(i) Let S_i be the closed set. Then,

$$\boxed{\bigcup_{i=1}^{N} S_i \text{ is also closed}}$$

where N is some finite number.

The finite collection of closed sets is closed.

(ii) Let S_i be closed. Then

$$\boxed{\bigcap_{i=1}^{N} S_i \text{ is closed}}$$

where N could be any number & may not be finite.

$\uparrow$

The intersection of any collection of closed sets is closed.

(iii) The union of any collection of open sets is open.

(iv) The intersection of a finite collection open sets is open.

(v) A set S is open if and only if all of its elements are interior points.

(vi) Every subspace of $\mathbb{R}^n$ is closed.

(vii) A set S is compact if and only if every sequence of elements of S has a subsequence that converges to an element of S.

(viii) A subset of $\mathbb{R}^n$ is compact if and only if it is closed & bounded.

→ ## Functions & Continuity

Let $X \subseteq \mathbb{R}^n$ be set. A function f (multivariate function) is a mapping from X to $\mathbb{R}^m$

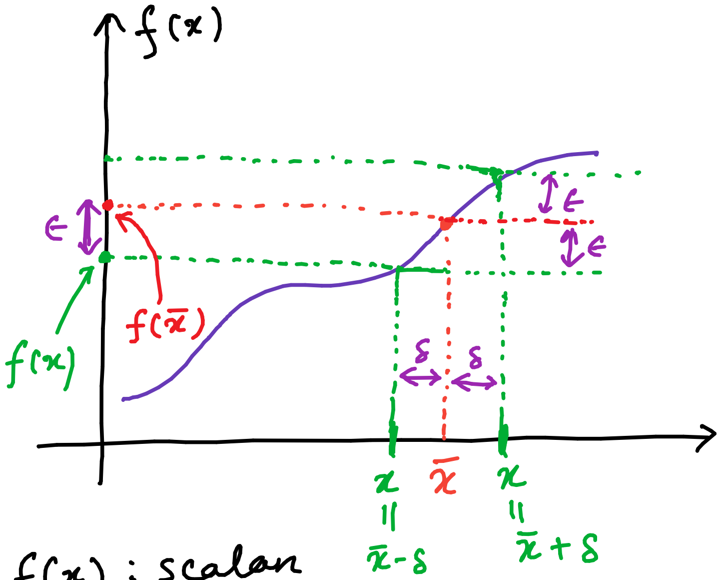
$$f : X \longrightarrow \mathbb{R}^m$$

Let $n = 2$ & $m = 3$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \qquad f(x) = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_1 \cdot x_2 \end{bmatrix} \in \mathbb{R}^3$$

- A function f is called "continuous at point $\bar{x} \in X$", if for every given $\epsilon > 0$, $\underbrace{\text{there exists}}_{\text{needs to find}}$ $\delta > 0$ such that

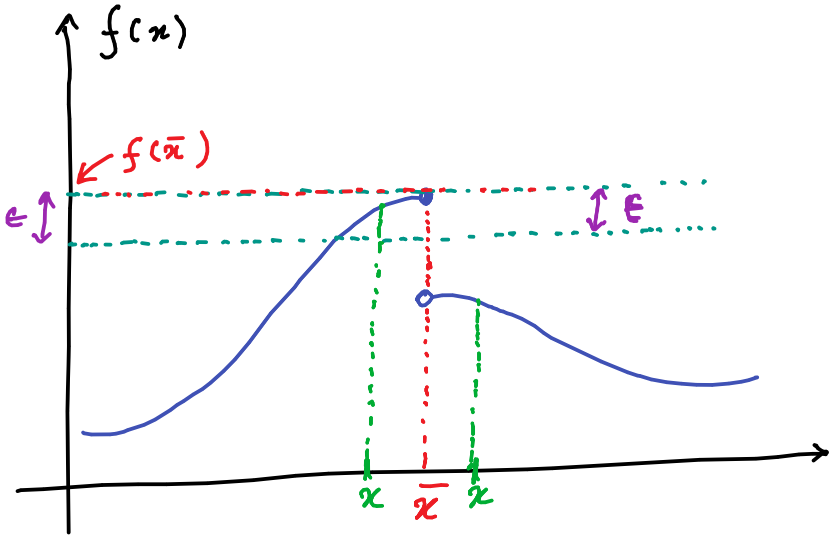
$$\|x - \bar{x}\| < \delta \Rightarrow \|f(x) - f(\bar{x})\| < \epsilon, \ \forall x \in X.$$



The function $f(x)$ in this figure is continuous at point $\bar{x}$, since for any given $\epsilon > 0$ one can always find δ s.t.

$f(x)$: scalar valued funct^n

$$|x - \bar{x}| < \delta \Rightarrow |f(x) - f(\bar{x})| < \epsilon$$



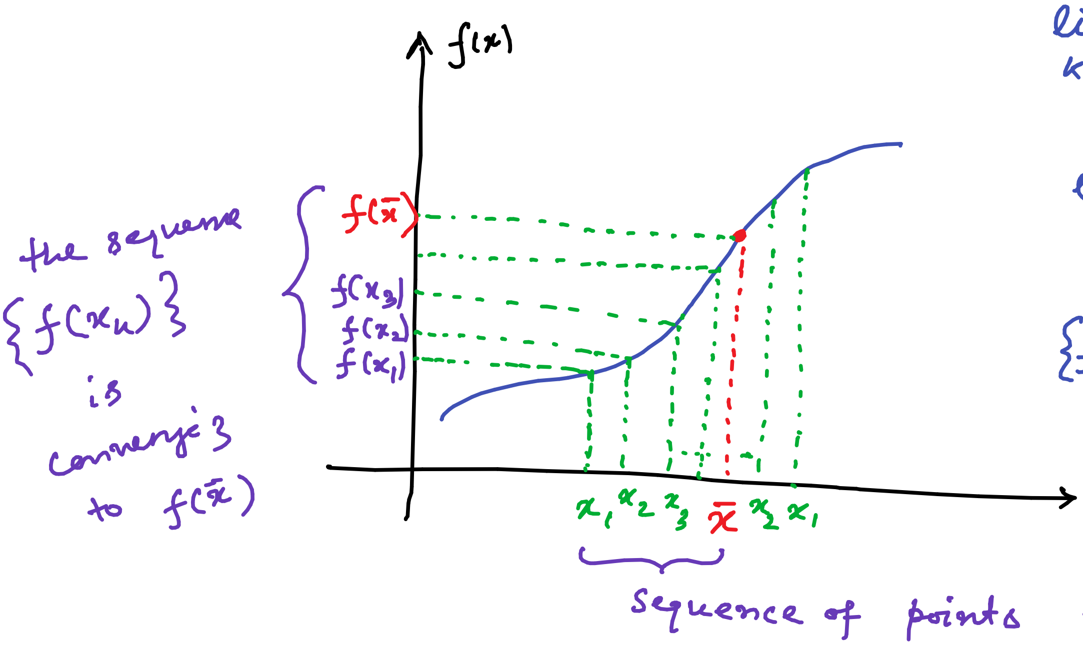
The function $f(x)$ here is discontinuous at point $\bar{x}$, since for the given $\epsilon > 0$ as shown here, we can not find a δ s.t.

whenever $|x - \bar{x}| < \delta \rightarrow |f(x) - f(\bar{x})| < \epsilon$

- Note here that the direction of choice of the points $x = \bar{x} + \delta$ or $x = \bar{x} - \delta$ is not specified. Hence, the ϵ, δ relation above must satisfy for all choice of x from any direction.

→ Another definition for continuity

- A function $f : X \to \mathbb{R}^m$, where $X \subseteq \mathbb{R}^n$, is continuous at point $\bar{x} \in X$, if for every sequence $\{x_k\}$ of points of X converging to $\bar{x}$, the sequence $\{f(x_k)\}$ converges to $f(\bar{x})$.



$$\lim_{k\to\infty} x_k = \bar{x}$$
$$\Downarrow$$
$$\lim_{k\to\infty} f(x_k) = f(\bar{x})$$
or
$$\{f(x_k)\} \to f(\bar{x})$$

- The function $f(x)$ is called continuous on $X \subseteq \mathbb{R}^n$ if it is continuous at every points in X.

→ A function $f : X \to \mathbb{R}^m$ with $X \subseteq \mathbb{R}$ is called :

- right-continuous at $\bar{x}$

$$\lim_{k\to\infty} x_k = \bar{x} \quad \text{with} \quad x_k \geq \bar{x} \quad \text{for all } k$$

$$\Downarrow$$

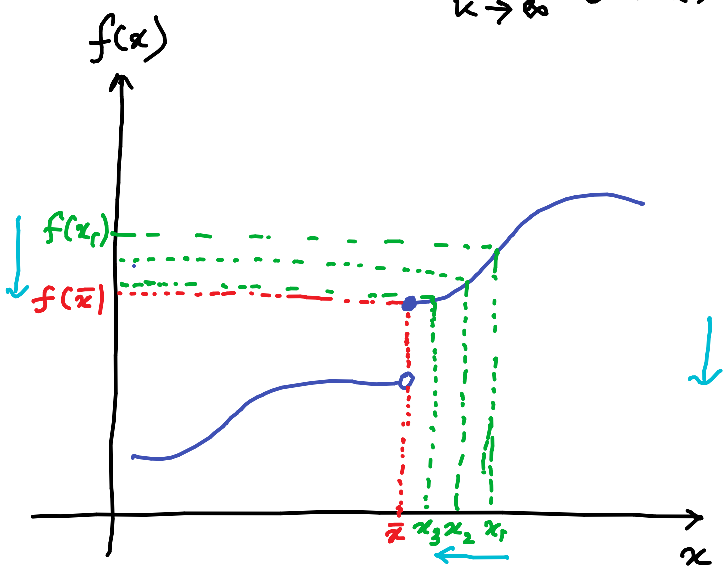
$$\lim_{k\to\infty} f(x_k) = f(\bar{x})$$

- left-continuous at $\bar{x}$

$$\lim_{k\to\infty} x_k = \bar{x} \quad \text{with} \quad x_k \leq \bar{x} \quad \text{for all}$$

$$\Downarrow$$

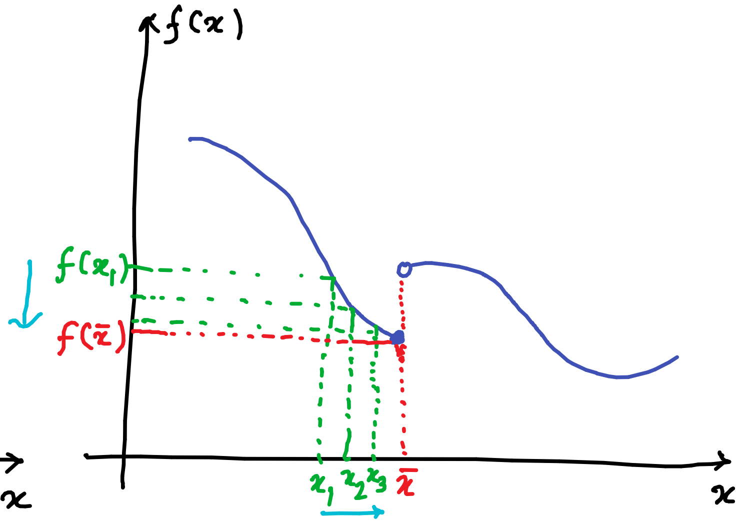
$$\lim_{k\to\infty} f(x_k) = f(\bar{x})$$



Right Continuous

All the sequence points x_k are right to $\bar{x}$

i.e. $\quad x_k \geq \bar{x}$ for all k



Left Continuous

All the sequence points x_k are chosen left to $\bar{x}$ i.e.

$x_k \leq \bar{x}$ for all k.

- A function f is continuous if and only if it is both left-continuous & right-continuous.

→ A realvalued function $f: X \to \mathbb{R}$ with $X \subseteq \mathbb{R}^n$, is called :

(i) lower semi-continuous at $\bar{x}$ if

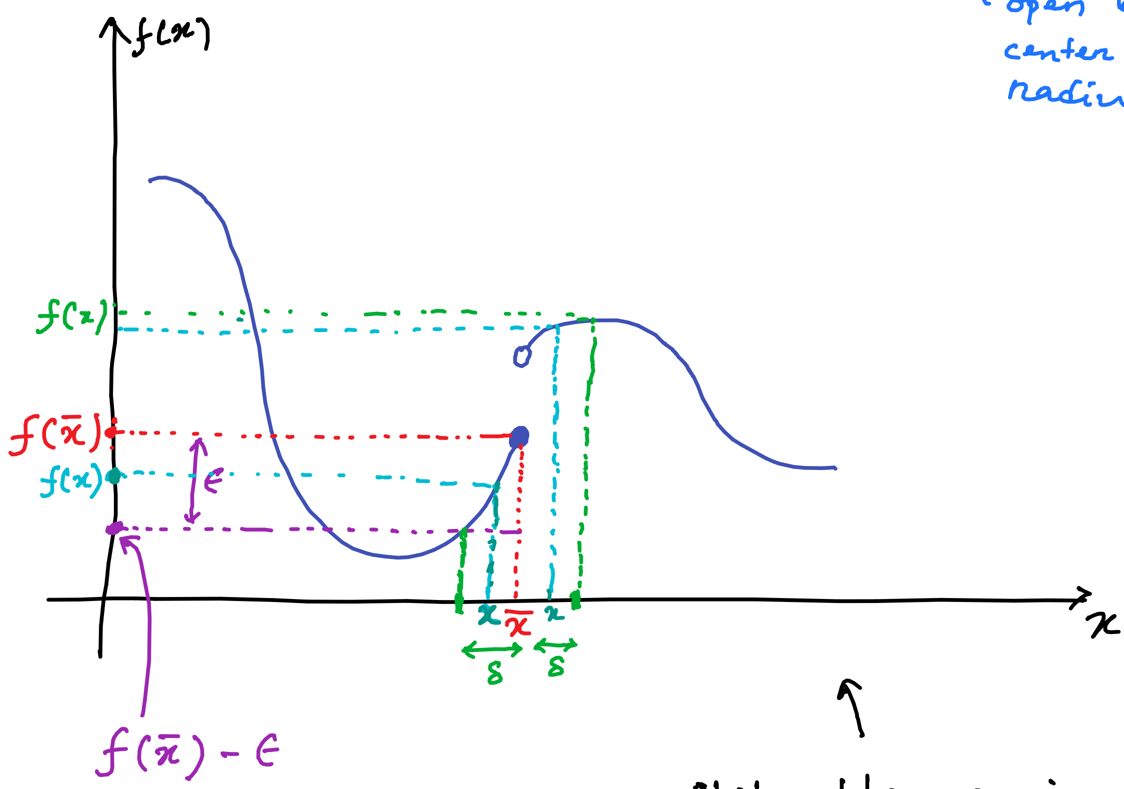
for every given $\epsilon > 0$, there exists $\delta > 0$ such that :

$$\boxed{\|x - \bar{x}\| < \delta \Rightarrow f(\bar{x}) - \epsilon < f(x),} \quad \forall x$$

$$|||$$

$$\boxed{f(\bar{x}) - \epsilon < f(x), \quad \forall x \in \mathcal{B}(\bar{x}, \delta)}$$

open ball center $\bar{x}$ & radius δ



Not upper-semi-continuous

- Upper semi-continuous at $\bar{x}$ if

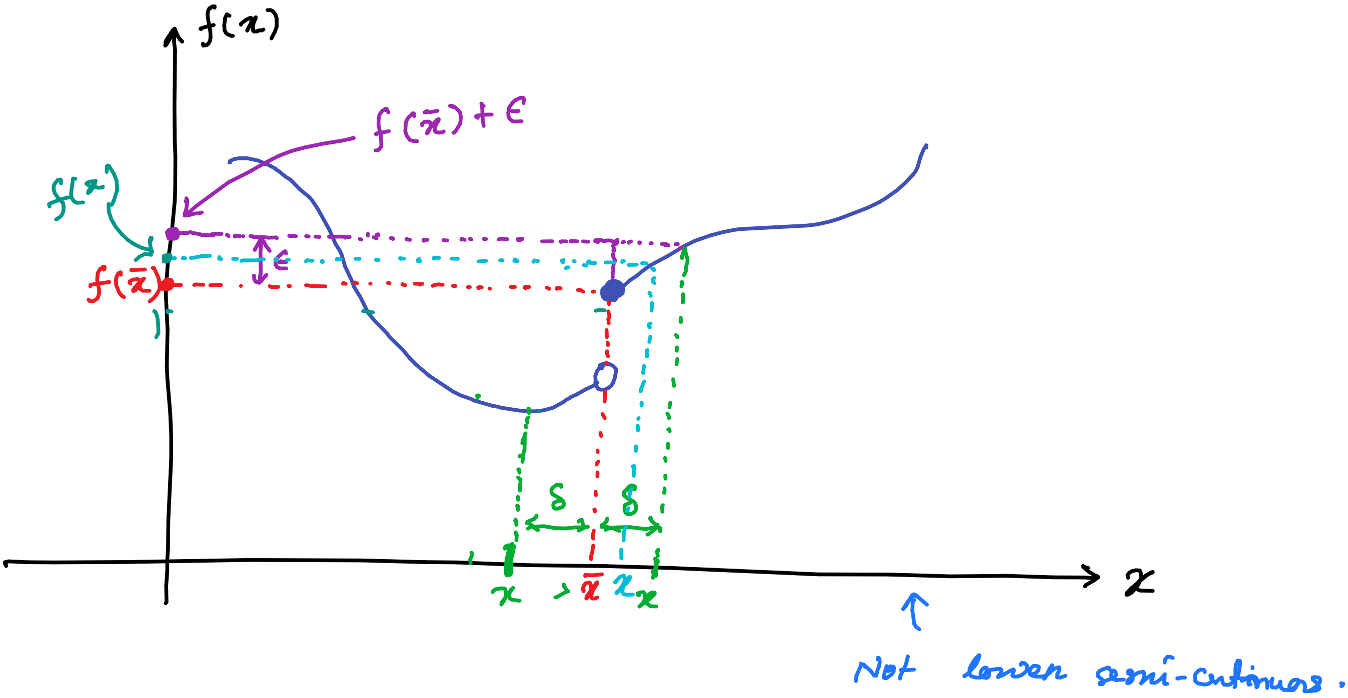
  for every given $\epsilon > 0$, there exists $\delta > 0$ (one can find)

  such that:

$$\boxed{\|x - \bar{x}\| < \delta \Rightarrow f(\bar{x}) + \epsilon > f(x) \quad \forall x}$$

$$\equiv$$

$$\boxed{f(\bar{x}) + \epsilon > f(x), \; \forall x \in \mathcal{B}(\bar{x}, \delta)}$$



→ For real valued function $f : X \to \mathbb{R}$ with $X \subseteq \mathbb{R}^n$

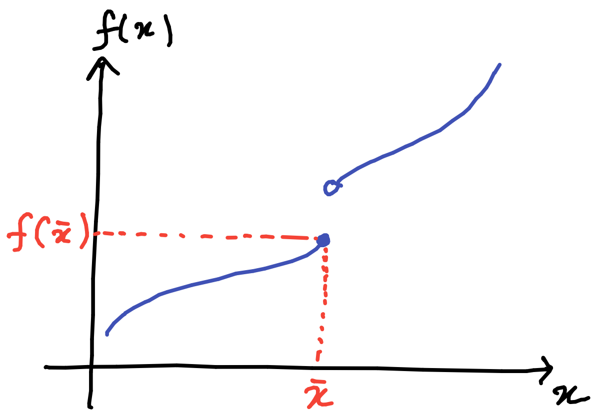
- lower semicontinuous if for every ϵ, $\exists\, \delta$ s.t.

$$\boxed{\begin{array}{c} \|x - \bar{x}\| < \delta \Rightarrow f(\bar{x}) - \epsilon < f(x) \\ \equiv \\ f(\bar{x}) - \epsilon < f(x), \; \forall x \in \mathcal{B}(\bar{x}, \delta) \end{array}}$$

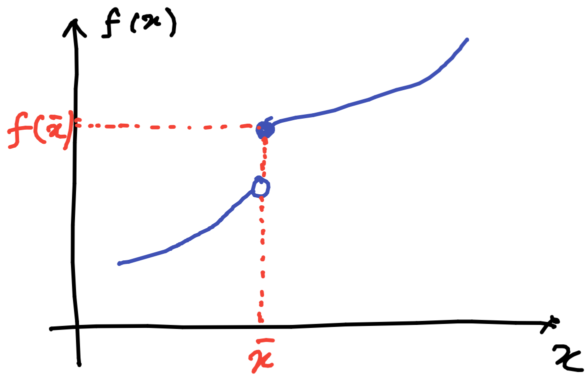
- upper semicontinuous if for every $\epsilon > 0$, $\exists\, \delta > 0$ s.t.

$$\boxed{\begin{array}{c} \|x - \bar{x}\| < \delta \Rightarrow f(\bar{x}) + \epsilon > f(x) \\ \equiv \\ f(\bar{x}) + \epsilon > f(x) \quad \forall x \in \mathcal{B}(\bar{x}, r) \end{array}}$$

- For a given continuous function f, one can thought of getting an upper semi-continuous funct^n (at say $\bar{x}$) by increasing the function value at point $\bar{x}$ to $f(\bar{x}) + \epsilon$ where $\epsilon > 0$. Similarly the lower semicontinuous function can by obtained by lowering the function value at $\bar{x}$ by $f(\bar{x}) - \epsilon$.



Lower Semicontinuous



Upper semicontinuous.

→ For a real-valued function $f : X \to \mathbb{R}$ with $X \subseteq \mathbb{R}$, if f is

- non-decreasing right-continuous

$$\Downarrow$$

f is upper semi-continuous

- non decreasing left-continuous

$$\Downarrow$$

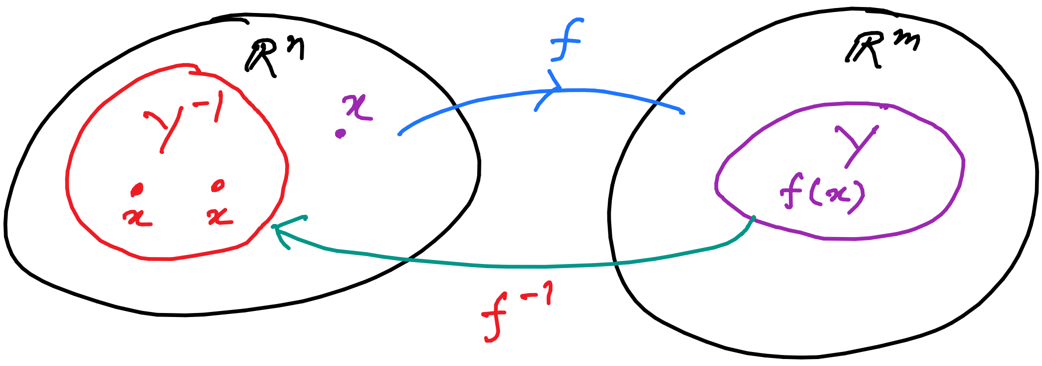
f is lower semi-continuous

→ Some Properties of Continuous functions

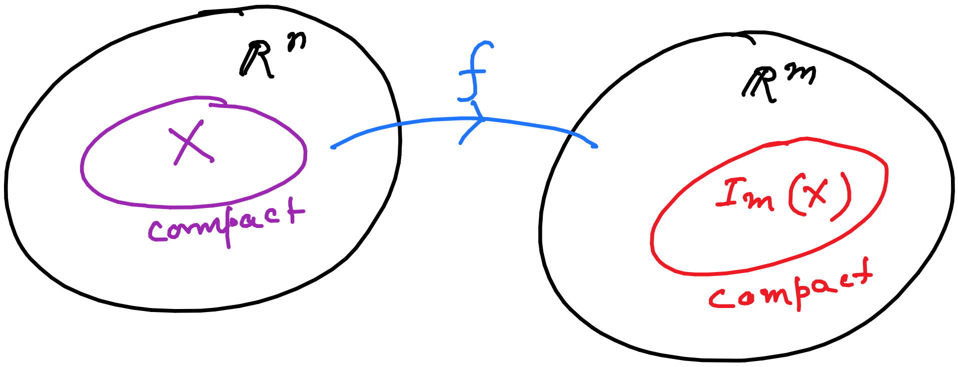
(i) Any vector norm on $\mathbb{R}^n$ is a continuous function.

(ii) Let $f : \mathbb{R}^m \to \mathbb{R}^p$ & $g : \mathbb{R}^n \to \mathbb{R}^m$ be continuous functions. Then, the composition: $f \circ g : \mathbb{R}^n \to \mathbb{R}^p$, defined as $f(g(x))$ is always a continuous function.

(iv) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous, & Y be an open subset (respectively, closed) subset of $\mathbb{R}^m$. Then the inverse image of $Y := \{x \in \mathbb{R}^n \mid f(x) \in Y\}$ is open (respectively, closed).



(iv) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Let $X \subseteq \mathbb{R}^n$ be compact. Then, the image of $X := \{ f(x) \mid x \in X \}$ is also a compact set.



Compactness property is preserved under mapping through continuous funct^{n}s.

(v) All affine functions $f(x) = Ax + b$ are continuous on entire $\mathbb{R}^n$.

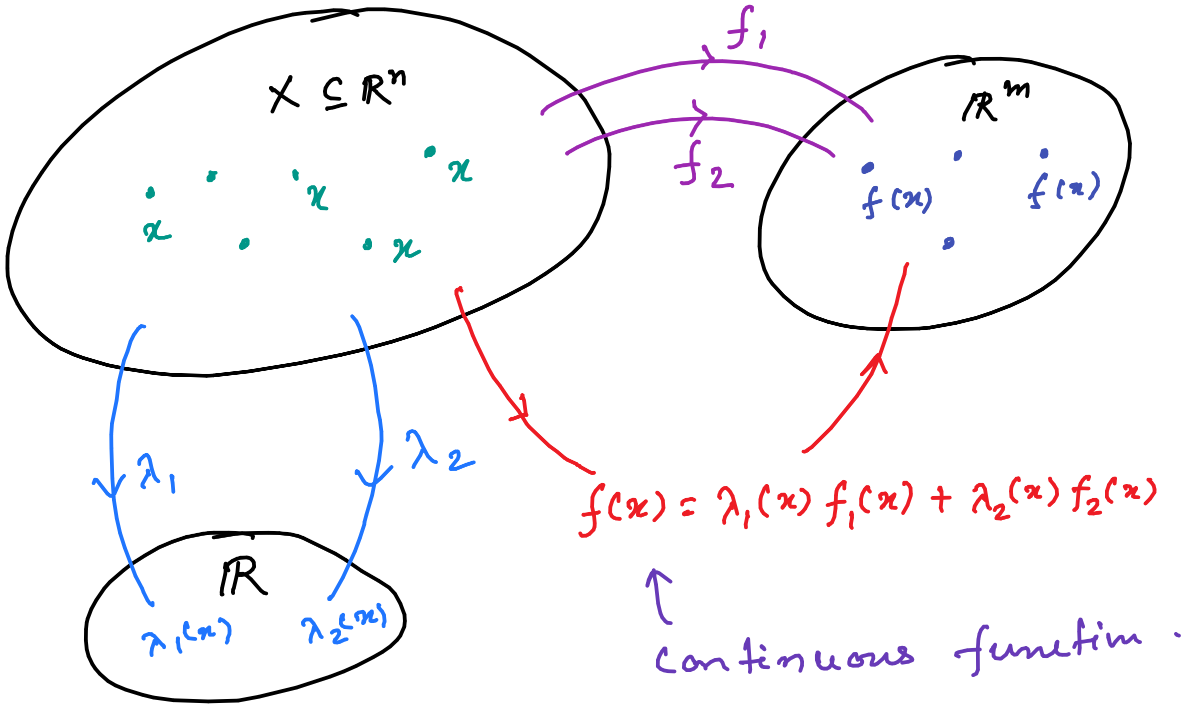
(vi) Let $f_1(x) : X \to \mathbb{R}^m$ & $f_2(x) : X \to \mathbb{R}^m$ with $X \subseteq \mathbb{R}^n$ be continuous functions.

Let $\lambda_1(x) : X \to \mathbb{R}$ & $\lambda_2(x) : X \to \mathbb{R}$ be two real-valued functions on X.

Then, the function:

$$f(x) = \lambda_1(x) f_1(x) + \lambda_2(x) f_2(x) \in \mathbb{R}^m$$

is continuous on X.



→ **Some Results**

Let X be a non-empty $\underbrace{\text{closed \& bounded}}_{}$ subset of $\mathbb{R}^n$. Then, following statements hold.

Compact

(i) If $f : X \to \mathbb{R}^m$ is continuous on X, then it is bounded on X, i.e. there exists a scalar $r > 0$ s.t.

$$\|f(x)\| \leq r, \quad \forall\ x \in X.$$

(ii) Let $f : X \to \mathbb{R}$ be a real-valued continuous function on X. Then, f attains its maximum at certain points in X. Similarly, f attains its minimum at certain points in X.

→ Weierstrass Result.

- The set of points $\bar{x} \in X$ where f attains its maximum is denoted as:

$$\boxed{\arg\max_{x \in X} f(x)}$$

Hence,

$$\bar{x} \in \arg\max_{x \in X} f(x)$$

often written as

$$\bar{x} = \arg\max_{x \in X} f(x)$$

- The set of points $\bar{x} \in X$ where f attains its minimum is denoted as:

$$\boxed{\arg\min_{x \in X} f(x)}$$

Hence

$$\bar{x} \in \arg\min_{x \in X} f(x)$$

$$\bar{x} = \arg\min_{x \in X} f(x)$$