

RESEARCH ARTICLE

Reduced Order Controller Synthesis with Regional Pole Constraint

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(Received 00 Month 200x; final version received 00 Month 200x)

A reduced order output feedback controller is designed for a linear time invariant system, which guarantees that the closed loop poles are placed within some pre-specified stability region in the complex plane. A convex approximation of the non-convex constraints is used to pose a sequence of semi-definite programs, which provide the lowest order proper controller satisfying the approximate constraints. The proposed method is demonstrated on two practical controller design applications.

Keywords: Linear systems, Output feedback, LMIs, Controller reduction

1 INTRODUCTION

The problem of finding minimum order output feedback controllers for various control objectives has proved to be difficult due to the underlying non-convexity of the optimizations involved (Bernstein 1992, Karimi et al. 2007, Hammer 1983). If all the closed loop poles are specified for an n^{th} order linear time invariant (LTI) single-input single-output (SISO) system, then it is well known that the minimum order output feedback controller which achieves these pole locations, is $(n - 1)$ (Wellstead 1991, L.Qiu and Zhou 2009). For the multi-input multi-output (MIMO) case, a minimum degree observer-controller configuration achieving arbitrary pole placement is given by the classic result due to Luenberger (Luenberger 1964). However, if there are no precise requirements on the closed loop poles, but they are only required to belong to some pre-specified region in the complex plane, then these extra degrees of freedom can be used to further reduce the controller order (e.g. below $(n - 1)$ in the SISO case). This pole placement scenario is more relevant in practice (Datta et al. 2012a,b, Datta and Chakraborty 2013, 2014) since the performance specifications usually mention time domain characteristics like settling time/damping ratio. Hence, it would be enough if a designed controller guarantees that all the closed loop poles are placed within some desirable region in the complex plane. Low order controllers, on the other hand, are usually desirable due to reduced implementation/computational complexities and related costs.

Under such a pole placement paradigm, in this article, we propose convex formulations to design a reduced order controller for general MIMO systems. The developed algorithms ensure that the resulting controller is a proper/strictly proper controller and the closed loop poles are placed inside a pre-specified region in the complex plane. The regional pole placement requirements on the closed loop poles are first translated into constraints in the coefficients of the corresponding polynomial matrices using the eliminant matrix (Antsaklis and Michel 2006). Thereafter this constraint set is convexified using a recent result in inner approximation of the polynomial matrix stability region (Henrion et al. 2003, Yang et al.

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This paper is partially supported by Department of Science and Technology, Government of India and Industrial Research and Consultancy Center, IIT Bombay.

2007). Finally, we show that a sequence of semi-definite programs (SDPs) has to be solved to obtain a reduced order proper controller for a strictly proper MIMO plant.

For SISO systems, we are able to pose and solve a slightly more general, partial pole placement problem. Frequently, one needs to exactly place a subset of the closed loop poles (which we call the critical poles) while the remaining poles (non-critical) can be placed anywhere within some pre-specified region. For example, in large interconnected power systems, the inter-area oscillations caused by electro-mechanical modes are a cause of concern for power system engineers and a typical controller would like to precisely place the poles corresponding to these oscillation modes. The other poles in the system are already quite stable and they can allowed to be placed anywhere within some pre-specified region corresponding to e.g. some settling time requirements. Such a pole placement paradigm was proposed in (Datta et al. 2012a) where simultaneously the state feedback controller norm was minimised. Here we use the same constructions as in (Datta et al. 2012a,b) along with a Sylvester parameterisation of output feedback controllers to optimise the controller order, while satisfying the separate requirements on the critical and non-critical poles. Similar to the MIMO case, we use an inner convex approximation of the polynomial stability region (Henrion et al. 2003, Yang et al. 2007) to define a linear matrix inequality (LMI) on the coefficients of the polynomials associated with the output feedback controller. It is shown that a reduced order proper output feedback controller, satisfying the approximated regional pole placement requirements, can be found by solving a sequence of SDPs.

The traditional approaches to find a reduced order controller for a linear system are based on various model or controller order reduction techniques (see e.g., Obinata and Anderson (2001) and the references therein). These methods provide no guarantees on the closed loop specifications and hence reduced order controller design with guaranteed closed loop performance, remains an important problem (Bernstein 1992). In Mesbahi and Papavassilopoulos (1997) and Mesbahi (1998), a similar problem to the one treated in this article, is posed as a rank minimization problem. It is shown that if the associated feasible set is a hyper-lattice, then it can be solved through an equivalent SDP. Similarly in Wang and Chow (2000), a convex suboptimal problem, associated with obtaining a reduced order controller, is solved by using the strictly positive real condition. In these approaches, convexification is achieved at the cost of optimality or some special system properties are assumed. In Keel and Bhattacharyya (1990), a reduced order controller is designed with regional pole placement requirements. An approximated output feedback controller is obtained through a non-convex iterative algorithm. This algorithm requires initial guess of a pseudo diagonal matrix consisting of eigenvalues taken from the pre-defined stability region and the controller gain matrix. Since the optimization uses Sylvester equation, one has to check before the start of each iteration that the eigenvalues of the pseudo diagonal matrix are not close to the open loop poles. Furthermore, this approach fails to place a subset of the eigenvalues at specific locations. On the other hand, the algorithm proposed in this paper requires one to heuristically choose certain polynomials which in turn determines the conservativeness of the solution obtained. From the implementation perspective, one of the requirements while designing a reduced order controller is that the computed controller should be proper or strictly proper. Typical approaches adopted in the literature are: i) representing the dynamics of plant as well as controller in a state-space form (Mesbahi and Papavassilopoulos (1997), Mesbahi (1998), Keel and Bhattacharyya (1990), Han et al. (2006)), ii) expressing the controller as a proper transfer function where the highest degree coefficient of denominator polynomial is set to one (Wang and Chow (2000)) and iii) imposing extra constraints in the optimizations (Han et al. (2006)). Another related problem is the design of fixed order controllers, where, Yang et al. (2007), Khatibi et al. (2008), Karimi et al. (2007), have focused on obtaining fixed order controllers for plants with polytopic uncertainty.

The rest of this paper is organized as follows. First we present some known definitions and results on polynomial matrices in Section 2. In Section 2.3 a procedure to construct the eliminant matrix associated with polynomial system matrices is reviewed. This matrix is used in Section 3 to synthesize a reduced order controller satisfying regional pole placement requirements. The partial pole placement problem for SISO systems is formulated in Section 4. In Section 5, the reduced order controller is obtained by solving at most n SDPs. Finally, case studies demonstrating the application of the proposed theory on a NASA F-8 DFBW aircraft and a 4-machine, 2-area power system, are included in Section 6.

2 Regional Pole Placement for MIMO Systems

We introduce some notations and definitions related to polynomial matrices before formulating the problem.

2.1 Preliminaries

Let us denote $\mathbb{R}[s]$ and $\mathbb{R}^{m \times p}[s]$ as the sets of all polynomials and $(m \times p)$ polynomial matrices respectively. Consider a polynomial matrix $A(s) \in \mathbb{R}^{m \times p}[s]$ with its entries $a_{ij}(s) \in \mathbb{R}[s]$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, p$. Let r be the highest degree occurring among the degrees of the polynomial entries $a_{ij}(s)$. Then the polynomial matrix $A(s)$ can be represented as

$$A(s) = A_r s^r + A_{r-1} s^{r-1} + \dots + A_1 s + A_0 \quad (1)$$

where $A_k \in \mathbb{R}^{m \times p}$ for $k = 0, 1, 2, \dots, r$ are the coefficient matrices of $A(s)$. Henceforth we will denote A_k as the k^{th} coefficient of $A(s)$ and r as the *degree* of the polynomial matrix (Wolovich 1974, Antsaklis and Michel 2006). The maximum degree occurring among the degrees of all elements in the j^{th} column, $a_j(s)$ of polynomial matrix $A(s)$, is referred to as the *column degree* of $a_j(s)$ and denoted as $\delta_c(a_j(s))$. Likewise the maximum degree occurring among the degrees of all elements in the i^{th} row, $a_i(s)$ of $A(s)$, is referred to as the *row degree* of $a_i(s)$ and denoted as $\delta_r(a_i(s))$. A polynomial matrix $A(s) \in \mathbb{R}^{m \times m}[s]$ is said to be \mathbb{S} -stable if all the zeros of $A(s)$ (i.e. roots of $\det A(s) = 0$, where \det denotes the determinant) belong to some stability region \mathbb{S} in the complex plane. Following Henrion et al. (2003), we will define \mathbb{S} as follows:

$$\mathbb{S} = \left\{ s \in \mathbb{C} : \begin{bmatrix} 1 & s^* \\ \underbrace{\begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}}_S \begin{bmatrix} 1 \\ s \end{bmatrix} < 0 \end{bmatrix} \right\} \quad (2)$$

where s^* denotes the complex conjugate of s and $S \in \mathbb{R}^{2 \times 2}$. It has been shown that this region \mathbb{S} can be used to represent some common stability regions in the complex plane (e.g. arbitrary half planes and discs (Henrion et al. 2003)).

Let us assume that the column degrees of a polynomial matrix $A(s) \in \mathbb{R}^{m \times m}[s]$ are $\delta_c(a_j(s)) = \mu_j$ for $j = 1, 2, \dots, m$. Then $A(s)$ can always be written as $A(s) = A_h P(s) + A_l(s)$ where $P(s) = \text{diag}\{s^{\mu_1}; s^{\mu_2}; \dots; s^{\mu_m}\}$, (where $\text{diag}\{\bullet\}$ denotes the diagonal matrix) $A_h \in \mathbb{R}^{m \times m}$ is the *highest column degree coefficient matrix* of $A(s)$ and $A_l(s)$ is the polynomial matrix consisting of remaining lower degree terms of $A(s)$. We say that $A(s)$ is *column reduced* if $\det A_h \neq 0$ (Kailath 1980, Wolovich 1974, Antsaklis and Michel 2006). Likewise, let $\delta_r(x_i(s)) = \nu_i$ for $i = 1, 2, \dots, m$ are the row degrees of a polynomial matrix $X(s) \in \mathbb{R}^{m \times m}[s]$. Then $X(s)$ can always be written as $X(s) = P(s)X_h + X_l(s)$ where $P(s) = \text{diag}\{s^{\nu_1}; s^{\nu_2}; \dots; s^{\nu_m}\}$. We say that $X(s)$ is *row reduced* if $\det X_h \neq 0$.

2.2 Problem Formulation

Let $H(s)$ be the transfer function matrix associated with a controllable and observable MIMO system with m inputs and p outputs. Then, it is well known (Kailath 1980, Wolovich 1974, Antsaklis and Michel 2006) that $H(s)$ can be represented by the following co-prime factorization:

$$H(s) = B(s)A(s)^{-1} \quad (3)$$

where $B(s) \in \mathbb{R}^{p \times m}[s]$ and $A(s) \in \mathbb{R}^{m \times m}[s]$ is column reduced. Let $\delta_c(a_j(s)) = \mu_j$ for $j = 1, 2, \dots, m$ and assume that $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$. If this is not the case then one has to perform suitable column operations

on $A(s)$ to make $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$. Note that, the same operations need to be performed on $B(s)$ to keep the transfer function matrix $H(s)$ same. Since $A(s)$ is column reduced we have (Antsaklis and Michel 2006)

$$\det A(s) = \det A_h s^{\sum \mu_j} + \text{lower degree terms}$$

and hence, the *order* of the plant is $n = \sum_{j=1}^m \mu_j$. Assume that the plant is strictly proper, that is, $\delta_c(b_j(s)) < \mu_j$ for $j = 1, 2, \dots, m$. Consider a controller $C(s)$, represented by the following factorization:

$$C(s) = X(s)^{-1}Y(s) \quad (4)$$

where $Y(s) \in \mathbb{R}^{m \times p}[s]$ and $X(s) \in \mathbb{R}^{m \times m}[s]$ is row reduced. By denoting $\delta_r(x_i(s)) = \nu_i$, we define the *order* of the controller as $\kappa := \sum_{i=1}^m \nu_i$.

It is well known that if the plant (3) and controller (4) are interconnected, then the closed loop poles are the zeros of the polynomial matrix

$$D(s) = X(s)A(s) + Y(s)B(s). \quad (5)$$

Then, the problem of interest can precisely be written as follows.

Problem 2.1 Find a minimum order ($\kappa \leq n$) proper/strictly proper controller $C(s)$ such that all the closed loop poles, that is, zeros of $D(s)$ are placed anywhere in the stability region \mathbb{S} .

In the following section we first introduce the eliminant matrix and then we show how that can be used to design a reduced order controller.

2.3 Eliminant Matrix

For any fixed integer $\nu > 0$, let us define a polynomial matrix $W^\nu(s) \in \mathbb{R}^{(n+m\nu) \times m}[s]$ as follows:

$$W^\nu(s) := \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{\mu_1+\nu-1} & & & \\ & 1 \\ & s \\ & \vdots \\ & s^{\mu_2+\nu-1} & & \\ & & \ddots & \\ & & & 1 \\ & & & s \\ & & & \vdots \\ & & & s^{\mu_m+\nu-1} \end{bmatrix}. \quad (6)$$

Then, corresponding to the polynomial matrices $A(s)$ and $B(s)$ of (3), for some integer $\nu > 0$, we can write

$$\begin{bmatrix} B(s) \\ sB(s) \\ \vdots \\ s^{\nu-1}B(s) \\ A(s) \\ sA(s) \\ \vdots \\ s^{\nu-1}A(s) \end{bmatrix} = M_\nu W^\nu(s). \quad (7)$$

In (7), we say the matrix $M_\nu \in \mathbb{R}^{\nu(p+m) \times (n+m\nu)}$ as the *eliminant matrix* associated with the polynomial matrices $A(s)$ and $B(s)$ (Wolovich 1974, Antsaklis and Michel 2006).

Next we assume that the degree of $X(s)$ and $Y(s)$ are both $\nu - 1$. This is reasonable since we do not know the relative degree of the controller a priori. Denote $X_k \in \mathbb{R}^{m \times m}$ and $Y_k \in \mathbb{R}^{m \times p}$ for $k = 0, 1, \dots, \nu - 1$ as the coefficient matrices associated with $X(s)$ and $Y(s)$ respectively. Then, by defining a controller coefficient matrix

$$\mathcal{K}_{\nu-1} := [Y_0 \ Y_1 \ \cdots \ Y_{\nu-1} \ X_0 \ X_1 \ \cdots \ X_{\nu-1}], \quad (8)$$

we can write

$$X(s)A(s) + Y(s)B(s) = \mathcal{K}_{\nu-1} \begin{bmatrix} B(s) \\ sB(s) \\ \vdots \\ s^{\nu-1}B(s) \\ A(s) \\ sA(s) \\ \vdots \\ s^{\nu-1}A(s) \end{bmatrix} = \mathcal{K}_{\nu-1} M_\nu W^\nu(s) \quad (9)$$

Since, $A(s)$ and $B(s)$ are right co-prime, it is known (Antsaklis and Michel 2006, Chapter 7, Theorem 2.13) that there exists some $\nu > 0$ such that the eliminant matrix is full column rank, that is, $\text{rank}(M_\nu) = n + m\nu$ where $n = \sum_{j=1}^m \mu_j$. Hence, for any arbitrary choice of $D(s) \in \mathbb{R}^{m \times m}[s]$ such that $\delta_c(d_j(s)) \leq \mu_j + \nu - 1$, we can solve for $X(s)$ and $Y(s)$ which satisfy the Diophantine equation (5). Furthermore, since $\delta_c(d_j(s)) \leq \mu_j + \nu - 1$, we have

$$D(s) = \mathbf{D}_t W^\nu(s) \quad (10)$$

where $\mathbf{D}_t \in \mathbb{R}^{m \times (n+m\nu)}$ (recall that column degrees of $W^\nu(s)$ are $\mu_j + \nu - 1$). Then, using (9) and (10), the Diophantine equation (5) will be satisfied if and only if following relation:

$$\mathcal{K}_{\nu-1} M_\nu = \mathbf{D}_t \quad (11)$$

holds (Antsaklis and Michel 2006). In the following section we use (11) to design a reduced order controller.

3 Reduced Order Controller Design

Recall that μ_1 was assumed to be the largest among all μ_j 's. Then, according to the definition (6), $W^v(s)$ can be written in the following form:

$$W^v(s) = W_0 + W_1s + W_2s^2 + \cdots + W_{\mu_1+v-1}s^{\mu_1+v-1}.$$

Next, using M_v , \mathbf{D}_t and $W^v(s)$, we construct two new matrices L_v and \mathbf{D} as follows:

$$\begin{aligned} L_v &= [L_0 \ L_1 \ \cdots \ L_{\mu_1+v-1}] \quad \text{where} \\ L_0 &= M_v W_0, \ L_1 = M_v W_1, \ \cdots, \ L_{\mu_1+v-1} = M_v W_{\mu_1+v-1} \\ \mathbf{D} &= [D_0 \ D_1 \ \cdots \ D_{\mu_1+v-1}] \quad \text{where} \\ D_0 &= \mathbf{D}_t W_0, \ D_1 = \mathbf{D}_t W_1, \ \cdots, \ D_{\mu_1+v-1} = \mathbf{D}_t W_{\mu_1+v-1} \end{aligned} \quad (12)$$

with $L_k \in \mathbb{R}^{v(p+m) \times m}$ and $D_k \in \mathbb{R}^{m \times m}$ for $k = 1, 2, \dots, \mu_1 + v - 1$. In the previous section we saw that the Diophantine equation (5) can be written as $\mathcal{H}_{v-1} M_v W^v(s) = \mathbf{D}_t W^v(s)$, and hence, by equating the coefficients of both sides we can write

$$\mathcal{H}_{v-1} L_v = \mathbf{D}. \quad (13)$$

The elements D_k 's in \mathbf{D} are the coefficients of the polynomial matrix $D(s)$. Since there exists some $v > 0$ such that (11) is solvable, the relation (13) also has a solution.

Recall that we are interested in designing a minimum order proper controller $C(s)$ such that the zeros of polynomial matrix $D(s)$ belong to some stability region \mathbb{S} . For this purpose, let us define a set

$$\begin{aligned} \mathcal{N}_s &:= \{D(s) \in \mathbb{R}^{m \times m}[s] \text{ of degree } \mu_1 + v - 1 : \\ &\quad \text{the zeros of } D(s) \in \mathbb{S}\} \end{aligned}$$

Let $q := \mu_1 + v - 1$. Hence Problem 2.1 can be posed as follows: find a minimum order proper controller $C(s)$ such that $D(s) \in \mathcal{N}_s$. However, it was shown (Henrion et al. 2003) that the set \mathcal{N}_s is a non-convex set. Hence, to convexify the optimization we use a result by Henrion et al. (Henrion et al. 2003) which is described briefly in the following section.

3.1 LMI Stability Region

Let us define following two matrices corresponding to the polynomial matrices $D(s)$ and an arbitrary but fixed $\bar{D}(s)$ (of degree q) respectively:

$$\begin{aligned} \mathbf{D} &:= [D_0 \ D_1 \ \cdots \ D_{q-1} \ D_q] \in \mathbb{R}^{m \times (q+1)m}; \\ \bar{\mathbf{D}} &:= [\bar{D}_0 \ \bar{D}_1 \ \cdots \ \bar{D}_{q-1} \ \bar{D}_q] \in \mathbb{R}^{m \times (q+1)m}. \end{aligned}$$

Then, for a fixed $\bar{D}(s) \in \mathcal{N}_s$, define the set:

$$\begin{aligned} \mathcal{M}_s &:= \{D(s) \in \mathbb{R}^{m \times m}[s] : \bar{\mathbf{D}}^T \mathbf{D} + \mathbf{D}^T \bar{\mathbf{D}} - \Pi^T (S \otimes T) \Pi \succ 0, \\ &\quad \text{for some } T = T^T \in \mathbb{R}^{qm \times qm}\} \end{aligned} \quad (14)$$

where \otimes refers to the Kronecker product, $\succ 0$ implies a positive definite matrix, S as defined in (2) and $\Pi \in \mathbb{R}^{2qm \times (q+1)m}$ denotes a projection matrix given by

$$\Pi = \begin{bmatrix} I_m & & 0 & \cdots & 0 \\ & \ddots & & & \\ & & I_m & & \\ & & & \ddots & \\ 0 & \cdots & 0 & & I_m \end{bmatrix}^T. \quad (15)$$

It was shown (Henrion et al. 2003, Lemma 1) that for any given \mathbb{S} -stable polynomial matrix $\bar{D}(s) \in \mathcal{N}_s$, the polynomial matrix $D(s) \in \mathcal{N}_s$ if there exists a symmetric matrix $T \in \mathbb{R}^{qm \times qm}$ satisfying the matrix inequality $\bar{\mathbf{D}}^T \mathbf{D} + \mathbf{D}^T \bar{\mathbf{D}} - \Pi^T (S \otimes T) \Pi \succ 0$. Hence, for every fixed $\bar{D}(s)$, this result characterizes a subset of the stable polynomial matrices and hence the set $\mathcal{M}_s \subseteq \mathcal{N}_s$. Then, by replacing the set \mathcal{N}_s with an approximated set \mathcal{M}_s we can pose a convexified but suboptimal version of Problem 2.1 as follows:

Problem 3.1 Find a minimum order ($\kappa \leq n$) proper controller $C(s)$ such that $D(s) \in \mathcal{M}_s$.

Note that, since we use a convex inner approximated set of the stable polynomial matrices, the optimization might not produce the minimum order controller, and hence we refer to the resulting controller as *reduced order controller*. However, the order of the resulting controller is minimum with respect to the approximated stability region.

3.2 LMI Formulation for Controller Design

Recall (13), i.e. $\mathcal{K}_{v-1} L_v = \mathbf{D}$. Then, the matrix inequality $\bar{\mathbf{D}}^T \mathbf{D} + \mathbf{D}^T \bar{\mathbf{D}} - \Pi^T (S \otimes T) \Pi \succ 0$, used to describe the set \mathcal{M}_s , would be

$$\bar{\mathbf{D}}^T \mathcal{K}_{v-1} L_v + L_v^T \mathcal{K}_{v-1}^T \bar{\mathbf{D}} + \Pi^T (S \otimes T) \Pi \succ 0. \quad (16)$$

For a given $\bar{\mathbf{D}}$, the inequality in (16) is linear in variables \mathcal{K}_{v-1} and T . Hence, we can use this as a constraint in the optimization problem. However, the solution of the LMI in (16) might not produce a row reduced polynomial matrix $X(s)$ and hence the resulting controller may not be a proper/strictly proper controller. Next, we propose a methodology to overcome this difficulty. Before proceeding further, let us introduce some more notations. Denote \mathbf{x}_i^T ($i = 1, 2, \dots, m$) as the i^{th} row of the matrix $X_{v-1} \in \mathbb{R}^{m \times m}$ which was defined as the highest degree coefficient of polynomial matrix $X(s)$. Then, construct a vector $\tilde{\mathbf{x}}_i^T \in \mathbb{R}^{m-1}$ by taking all the elements of \mathbf{x}_i^T excluding x_{ii} for $i = 1, 2, \dots, m$. Denote the elements of $\tilde{\mathbf{x}}_i^T$ as \tilde{x}_{ik} for $k = 1, 2, \dots, m$ and $i \neq k$. Then, the following result holds.

Theorem 3.2: For a fixed polynomial matrix $\bar{D}(s) \in \mathcal{N}_s$, if \mathcal{K}_{v-1} and a symmetric matrix T , satisfy the following conditions:

$$(i) \quad \bar{\mathbf{D}}^T \mathcal{K}_{v-1} L_v + L_v^T \mathcal{K}_{v-1}^T \bar{\mathbf{D}} + \Pi^T (S \otimes T) \Pi \succ 0$$

$$(ii) \quad \begin{bmatrix} 1 & \\ & (m-1)x_{ii}I_{m-1} & \tilde{\mathbf{x}}_i \\ & \tilde{\mathbf{x}}_i^T & x_{ii} \end{bmatrix} \succ 0 \text{ for } i = 1, 2, \dots, m,$$

then all the closed loop poles are placed within the stability region \mathbb{S} . Furthermore, the resulting controller would be either proper or strictly proper and the order of the controller $\kappa = \sum_{i=1}^m \delta_r(x_i(s))$.

Proof For a fixed $\bar{\mathbf{D}}$, since the matrices \mathcal{K}_{v-1} and T are satisfying condition (i), we have $\bar{\mathbf{D}}^T \mathbf{D} + \mathbf{D}^T \bar{\mathbf{D}} - \Pi^T (S \otimes T) \Pi \succ 0$. Hence, the polynomial matrix $D(s) \in \mathcal{M}_s$. However, following the previous discussion, we have $\mathcal{M}_s \subseteq \mathcal{N}_s$ and hence all the closed loop poles belong to the stability region \mathbb{S} . According to the

Schur complement relation, the condition (ii) is equivalent to

$$x_{ii} > 0 \text{ and } x_{ii}^2 - (m - 1)\tilde{\mathbf{x}}_i^T \tilde{\mathbf{x}}_i > 0,$$

and hence we can write

$$\begin{aligned} x_{ii}^2 &> (m - 1)\tilde{\mathbf{x}}_i^T \tilde{\mathbf{x}}_i \text{ for } i = 1, 2, \dots, m \\ &= (m - 1) \sum_{k=1}^m |\tilde{x}_{ik}|^2 \text{ for } i = 1, 2, \dots, m \text{ and } i \neq k \\ &= \sum_{k=1}^m |\tilde{x}_{ik}|^2 + (m - 2) \left(\sum_{k=1}^m |\tilde{x}_{ik}|^2 \right) \\ &\geq \sum_{k=1}^m |\tilde{x}_{ik}|^2 + 2 \left(\sum_{l=1}^m \sum_{k=1}^m |x_{ik}| |x_{il}| \right) \text{ for } i \neq k \neq l \quad (\star) \\ &= \left(\sum_{k=1}^m |\tilde{x}_{ik}| \right)^2 \\ \Rightarrow |x_{ii}| &> \sum_{k=1}^m |\tilde{x}_{ik}| \text{ for } i \neq k \end{aligned} \tag{17}$$

In the above, the inequality at step (\star) follows from the Young's inequality¹. According to (17), the matrix X_{v-1} is strictly diagonally dominant matrix and hence is nonsingular (Horn and Johnson 1985). In addition, none of the diagonal entries are zero. Hence, according to the definition, the polynomial matrix $X(s)$ is row reduced with $\delta_r(x_i(s)) = v - 1$. Hence, we can write (Antsaklis and Michel 2006)

$$\det X(s) = \det X_{v-1} s^{mv-m} + \text{lower degree terms.}$$

This leads to the conclusion that inverse of the polynomial matrix $X(s)$ exists. In addition, following the construction of the controller coefficient matrix \mathcal{K}_{v-1} (see (8)), we have $\delta_r(x_i(s)) \geq \delta_r(y_i(s))$. Hence the resulting controller $C(s) = X(s)^{-1}Y(s)$ is either proper or strictly proper. The polynomial matrix $X(s)$ being row reduced, leads to the fact that the order of the controller is equal to $\sum_{i=1}^m \delta_r(x_i(s))$. This completes the proof. \square

3.3 Synthesis Procedure

Note that according to Theorem 3.2, if conditions (i) and (ii) are satisfied then we achieve our objectives: regional pole placement requirement with proper/strictly proper controller. Since we are interested in designing a minimum order proper controller $C(s)$, we can check the satisfiability of conditions (i) and (ii) starting with a zeroth order controller, that is, $v = 1$. If there is no feasible solution then the value can be sequentially increased until a feasible solution is reached. The satisfiability conditions can be checked by formulating the following LMI optimization problem:

Problem 3.3 Find $\max_{\mathcal{K}_{v-1}, T, \gamma} \gamma$ subject to

$$\begin{aligned} (i) \quad & \bar{\mathbf{D}}^T \mathcal{K}_{v-1} L_v + L_v^T \mathcal{K}_{v-1}^T \bar{\mathbf{D}} + \Pi^T (S \otimes T) \Pi - \gamma I \succ 0 \\ (ii) \quad & \begin{bmatrix} 1 & & & \\ & x_{ii} I & \tilde{\mathbf{x}}_i & \\ & \tilde{\mathbf{x}}_i^T & & \\ & & & x_{ii} \end{bmatrix} \succ 0 \text{ for } i = 1, 2, \dots, m \end{aligned}$$

¹Let x_1 and x_2 be two non-negative numbers. Then, according to Young's inequality we have $2x_1 x_2 \leq x_1^2 + x_2^2$.

where γ is a positive scalar and I denotes the identity matrix of appropriate dimension.

Since $A(s)$ and $B(s)$ are co-prime it is guaranteed that there exists some v such that the constraint (i) of the above problem is satisfiable. The role of constraint (ii) is explained below. Note that Problem 3.3 is an LMI optimization and hence can be solved by standard LMI solvers like *SeDuMi*. However, to solve Problem 3.3 we need to first choose a central polynomial matrix $\bar{D}(s) \in \mathcal{N}_S$. At the current state of research, the polynomial matrix $\bar{D}(s)$ has to be chosen heuristically. In the numerical example below we choose a diagonal $\bar{D}(s)$.

Remark 1: Note that the constraint (ii) in Problem 3.3 is required to guarantee that the resulting $X(s)$ is row reduced. However, according to the definition of row reduced polynomial matrix, the highest row degree coefficient matrix X_h should be nonsingular. Since it is difficult to write that as an LMI, a sufficient condition, i.e., X_{v-1} should be nonsingular, is imposed in Problem 3.3. From our experience with numerical examples, it seems that the solution of $\max_{\mathcal{N}_{v-1}, T, \gamma} \gamma$ with only constraint (i) of Problem 3.3, usually results in a nonsingular X_h . Hence it is preferable that Problem 3.3 should be first solved without the possibly conservative constraint (ii). If a feasible solution does not result in a nonsingular X_h then we need to impose constraint (ii) in the optimization.

Notice that if we impose constraint (ii) in Problem 3.3 then the highest row degree coefficient matrix $X_h = X_{v-1}$, as a result the order of the controller would be $\sum_{i=1}^m \delta_r(x_i(s)) = m(v-1)$. On the other hand, if the optimization is solved without considering constraint (ii) and results in a non singular X_h then X_h might not be equal to X_{v-1} , and hence the controller order could be $\sum_{i=1}^m \delta_r(x_i(s)) \leq m(v-1)$.

Summarizing the above, we propose following design procedure to obtain a reduced order controller:

Design Steps

- (1) Start with $v = 1$.
- (2) Following the procedure proposed in Section 2.3, compute the eliminant matrix M_v and construct L_v .
- (3) Choose a stability region \mathbb{S} in the complex plane. Design a central polynomial matrix $\bar{D}(s)$ such that all the zeros of $\bar{D}(s)$ are within \mathbb{S} . Some trial and error iterations are required at this stage. Then, Solve Problem 3.3 without considering constraint (ii). If the feasible solution results in a row reduced $X(s)$ then stop; otherwise solve Problem 3.3. If there is no feasible solution to Problem 3.3 (without considering constraint (ii)), then go to Step-4.
- (4) Increase the value of v by one and follow the procedure from Step-2. The increment of v should be continued until a feasible solution is achieved.

Remark 2: Although there is no unique strategy for the choice of \mathbb{S} -stable polynomial matrix $\bar{D}(s)$, we have observed through several practical examples that a diagonal polynomial matrix where the diagonal elements are constructed by considering all the well damped/stable open loop poles (which are inside the stability region \mathbb{S}) produces satisfactory results. Since in the applications, most of the open loop poles are stable and well damped, it is preferable to consider them while constructing the diagonal polynomials of $\bar{D}(s)$. However, if there are not enough well damped/stable open loop poles then we need to choose, heuristically, complex numbers (with conjugate) which are inside the stability region \mathbb{S} in the complex plane to form diagonal polynomials of $\bar{D}(s)$. We have experienced from the numerical examples that the choice of complex numbers near to the boundary of \mathbb{S} produces satisfactory results. More details on designing $\bar{D}(s)$ are included in the numerical examples.

4 Partial Pole Placement for SISO Systems

In this part we specialise the above controller order reduction technique for SISO system, so as to address the partial pole placement paradigm (Datta et al. 2012a, Datta and Chakraborty 2013) described in the introduction. In particular we consider the following constraints: (1) an arbitrary subset (critical poles) of the closed loop poles are to be placed at precise pre-defined locations in the complex plane, and (ii) the remaining (non-critical) poles are to be placed within a pre-defined region in the complex plane.

Consider an LTI SISO system represented by the following transfer function.

$$P(s) = \frac{b(s)}{a(s)} = \frac{b_{n-1}s^{n-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0} \quad (18)$$

where the polynomials $a(s)$ and $b(s)$ are co-prime. Assume that the plant $P(s)$ is strictly proper. Let us consider an output feedback controller of the following form:

$$C(s) = \frac{y(s)}{x(s)} = \frac{y_ms^m + y_{n-1}s^{n-1} + \dots + y_1s + y_0}{x_ms^m + x_{m-1}s^{m-1} + \dots + x_1s + x_0} \quad (19)$$

with $x_m \neq 0$ and $m \leq (n-1)$. The closed loop, comprising of plant $P(s)$ and controller $C(s)$, would be

$$G(s) = \frac{P(s)}{1 + P(s)C(s)} = \frac{b(s)x(s)}{a(s)x(s) + b(s)y(s)}, \quad (20)$$

and hence the corresponding characteristic polynomial is

$$\sigma(s) = a(s)x(s) + b(s)y(s) \quad (21)$$

with degree $(n+m)$.

It is well known (Wellstead 1991, L.Qiu and Zhou 2009), that if all the $(n+m)$ poles of the closed loop system are specified, then the minimum order of the required controller is $m = n-1$. However, here only a subset of the closed loop poles are specified while remaining are free. Assume q out of $(n+m)$ closed loop poles to be non-critical and hence not associated with any desired closed loop location. The remaining $(n+m-q)$ poles are critical and are required to be placed at self-conjugate locations $\{-\lambda_1, -\lambda_2, \dots, -\lambda_{n+m-q}\}$ in closed loop. Further assume that the q free poles are required to be located inside a (stable) subset \mathbb{S} (see (2)) of the complex plane \mathbb{C} . Then consider the following problem:

Problem 4.1 Find a minimum order ($m \leq (n-1)$) proper controller $C(s)$ such that the closed loop poles have the following properties:

- (1) $(n+m-q)$ out of the total $(n+m)$ poles are placed at $\{-\lambda_1, -\lambda_2, \dots, -\lambda_{n+m-q}\}$ and
- (2) remaining q poles are placed anywhere in \mathbb{S} .

Denote $\{-\mu_1, -\mu_2, \dots, -\mu_q\}$ as the unspecified q closed loop poles of the system (20). Hence the characteristic equation of the closed loop system will be

$$\sigma(s) = \underbrace{\left[\prod_{j=1}^q (s + \mu_j) \right]}_{\alpha(s)} \underbrace{\left[\prod_{i=1}^{n+m-q} (s + \lambda_i) \right]}_{\beta(s)} \quad (22)$$

where

$$\alpha(s) := s^q + \alpha_{q-1}s^{q-1} + \dots + \alpha_1s + \alpha_0; \quad \beta(s) := s^{n+m-q} + \beta_{n+m-q-1}s^{n+m-q-1} + \dots + \beta_1s + \beta_0.$$

Note that in (22), $\beta(s)$ is a monic polynomial of known coefficients (completely defined from the problem specifications) while $\alpha(s)$ is a monic polynomial of unknown coefficients. According to Problem 4.1, the only requirement on $\alpha(s)$ is that the roots should be located in a pre-specified region $\mathbb{S} \subset \mathbb{C}$ defined in (2). Next, denote the set of all q^{th} degree monic polynomials with real coefficients as $\mathbb{R}[s]$, and define the set $C_s := \{\alpha(s) \in \mathbb{R}[s] : \text{roots of } \alpha(s) \in \mathbb{S}\}$. Then, the second constraint of Problem 4.1 can be restated as $\alpha(s) \in C_s$. As pointed out in the MIMO case, the set $C_s \subset \mathbb{R}[s]$ is not a convex set for $q \geq 3$ (see (Ackermann 1980, Henrion et al. 2003)) which leads to a non-convex optimization problem. We

convexify this constraint using a SISO version of the technique described in Section 3.1 above (Henrion et al. 2003, Yang et al. 2007).

Assume that $\widehat{\alpha}(s)$ is a polynomial in the stability region C_s . Define the coefficient vectors corresponding to $\widehat{\alpha}(s)$ and $\alpha(s)$ (defined in (22)) as follows: $\widehat{\alpha} := [\widehat{\alpha}_0 \widehat{\alpha}_1 \dots \widehat{\alpha}_{q-1}]^T \in \mathbb{R}^q$; and $\alpha := [\alpha_0 \alpha_1 \dots \alpha_{q-1}]^T \in \mathbb{R}^q$ respectively. Further, let $\alpha_e := [\alpha^T \ 1]^T \in \mathbb{R}^{q+1}$ and $\widehat{\alpha}_e := [\widehat{\alpha}^T \ 1]^T \in \mathbb{R}^{q+1}$. A restatement of (14) above follows: for any given stable polynomial $\widehat{\alpha}(s) \in C_s$, the polynomial $\alpha(s)$ is also in C_s provided there exists a symmetric matrix $P \in \mathbb{R}^{q \times q}$ satisfying the matrix inequality

$$\alpha_e \widehat{\alpha}_e^T + \widehat{\alpha}_e \alpha_e^T - \Pi^T (S \otimes P) \Pi \succeq 0. \quad (23)$$

where Π is defined in (15) with $m = 1$. This helps us to convexify Problem 4.1 by replacing constraint (2) with (23), leading to a suboptimal (but convex) version.

However as in the matrix case, to compute (23) explicitly we still need *a priori* the central polynomial $\widehat{\alpha}(s) \in C_s$. In Henrion et al. (2003) and Yang et al. (2007), various domain dependent heuristics are provided for design choices for $\widehat{\alpha}(s)$. In our case $\widehat{\alpha}(s)$ can be chosen to be any q^{th} degree polynomial with roots in the stability region \mathbb{S} . Note that the accuracy of the approximation is sensitive to the choice of the central polynomial $\widehat{\alpha}(s)$ (see Henrion et al. (2003), Yang et al. (2007)), and hence some conservativeness is introduced in to the proposed methodology due to this dependence.

5 Controller Synthesis for SISO systems

In this section we show that a sub-optimal solution to Problem 4.1 can be obtained from an SDP. Define the following Toeplitz matrices corresponding to the polynomials $a(s)$ and $b(s)$ in (18):

$$T(a) := \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \\ 1 & a_{n-1} & \cdots & a_1 \\ 0 & 1 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad T(b) := \begin{bmatrix} b_0 & 0 & \cdots & 0 \\ b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-2} & b_{n-3} & \cdots & b_0 \\ b_{n-1} & b_{n-2} & \cdots & b_1 \\ 0 & b_{n-1} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{n-1} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad (24)$$

The Sylvester's resultant matrix associated with $T(a) \in \mathbb{R}^{(n+m+1) \times (m+1)}$ and $T(b) \in \mathbb{R}^{(n+m+1) \times (m+1)}$ can be defined as follows:

$$\mathcal{R}(a, b, 2(m+1)) := [T(a) \ T(b)]_{(n+m+1) \times 2(m+1)} \quad (25)$$

Let us define the vector $\sigma := [\sigma_0 \ \sigma_1 \ \cdots \ \sigma_{n+m-1} \ \sigma_{n+m}]^T \in \mathbb{R}^{n+m+1}$ associated with the closed loop characteristic polynomial $\sigma(s) := \sigma_{n+m}s^{n+m} + \cdots + \sigma_1 s + \sigma_0$. Further, define following vectors

$$x := [x_0 \ x_1 \ \cdots \ x_m]^T \in \mathbb{R}^{(m+1)} \quad \text{and} \quad y := [y_0 \ y_1 \ \cdots \ y_m]^T \in \mathbb{R}^{(m+1)} \quad (26)$$

corresponding to the polynomials $x(s)$ and $y(s)$ defined in (19). The controller coefficient vector can then be defined as follows: $k := [x \ y]^T \in \mathbb{R}^{2(m+1)}$.

Arbitrary pole placement with the controller $C(s)$ can be achieved (e.g. see Wellstead (1991), L.Qiu and Zhou (2009)) from the following relation:

$$[\mathcal{R}(a, b, 2(m+1))]k = \sigma \quad (27)$$

From (27) it can be verified that when $m = n - 1$ the matrix $\mathcal{R}(a, b, 2(m + 1))$ is square and also non-singular ($a(s)$ and $b(s)$ are co-prime). Hence there is a unique controller coefficient vector k corresponding to the specified σ . However, in our case, only a subset of the closed loop poles are specified, which leads to the following result:

Lemma 5.1: *For a given set of closed loop poles $\{-\lambda_1, -\lambda_2, \dots, -\lambda_{n+m-q}\}$, (27) defines the following linear equations*

$$\alpha = Fk + g \quad \text{and} \quad \tilde{E}k + \tilde{h} = \mathbf{0}. \tag{28}$$

for some $F \in \mathbb{R}^{q \times (2m+2)}$, $g \in \mathbb{R}^q$, $\tilde{E} \in \mathbb{R}^{(n+m+1-q) \times (2m+2)}$, $\tilde{h} \in \mathbb{R}^{(n+m+1-q)}$ and $\mathbf{0}$ is a zero vector of appropriate dimension.

Proof Following Datta et al. (2012a), the next $(n + m + 1)$ linear equations can be derived from (27) and (22)

$$\begin{aligned} a_0x_0 + b_0y_0 &= \beta_0\alpha_0 \\ a_1x_0 + a_0x_1 + b_1y_0 + b_0y_1 &= \beta_0\alpha_1 + \beta_1\alpha_0 \\ &\vdots \\ x_{m-1} + a_{n-1}x_m + b_{n-1}y_m &= \beta_{n+m-q-1}\alpha_{q-1} \\ x_m &= 1 \end{aligned} \tag{29}$$

From (29), it is possible to express α_j ($j = 0, 1, \dots, q - 1$) in terms of variables x_i 's and y_i 's ($i = 0, 1, \dots, m$). Compactly this can be written as $\alpha = Fk + g$ where $F \in \mathbb{R}^{q \times (2m+2)}$ and $g \in \mathbb{R}^q$.

Now, excluding $x_m = 1$, the coefficients $\alpha_0, \dots, \alpha_{q-1}$ can be back-substituted in the set of $(n + m)$ equations (29) to get $(n + m - q)$ linear equations in x_i 's and y_i 's (for $i = 0, 1, \dots, m$). These equations can be written in the form: $Ek + h = \mathbf{0}$ where $E \in \mathbb{R}^{(n+m-q) \times (2m+2)}$, $h \in \mathbb{R}^{(n+m-q)}$ and $\mathbf{0}$ is a zero vector of appropriate dimension. Including the equation $x_m - 1 = 0$ to the above set of equations, $Ek + h = \mathbf{0}$ can be written as $\tilde{E}k + \tilde{h} = \mathbf{0}$. \square

Corresponding to the relation $\alpha = Fk + g$, of Lemma 5.1, let us define α_e as

$$\alpha_e = \tilde{F}k + \tilde{g} \quad \text{where} \quad \tilde{F} = \begin{bmatrix} F_{q \times (2m+2)} \\ \mathbf{0}_{1 \times (2m+2)} \end{bmatrix} \quad \text{and} \quad \tilde{g} = \begin{bmatrix} g \\ 1 \end{bmatrix} \tag{30}$$

Using (30), the LMI (23) becomes

$$\tilde{F}k\hat{\alpha}_e^T + \hat{\alpha}_ek^T\tilde{F}^T + \tilde{g}\hat{\alpha}_e^T + \hat{\alpha}_e\tilde{g}^T - \Pi^T(S \otimes P)\Pi \succeq 0 \tag{31}$$

Then the following result holds:

Theorem 5.2: *For any fixed $\hat{\alpha}(s) \in C_s$, if for some $k \in \mathbb{R}^{2m+2}$ and for some $P = P^T \in \mathbb{R}^{q \times q}$, the relations (31) and $\tilde{E}k + \tilde{h} = \mathbf{0}$ hold, then the closed loop poles (roots of the polynomial defined in (21)) satisfy the following properties:*

- (1) $(n + m - q)$ out of the total $(n + m)$ poles are $\{-\lambda_1, -\lambda_2, \dots, -\lambda_{n+m-q}\}$.
- (2) the remaining q poles $-\mu_i \in \mathbb{S}$ for $i = 1, \dots, q$.

Furthermore, the resulting controller will be an m^{th} order proper or strictly proper controller.

Proof Fix $\hat{\alpha}(s) \in C_s$. Assume that some $k \in \mathbb{R}^{(2m+2)}$ and $P = P^T \in \mathbb{R}^{q \times q}$ satisfy (31). Then

$$\alpha_e\hat{\alpha}_e^T + \hat{\alpha}_e\alpha_e^T - \Pi^T(S \otimes P)\Pi \succeq 0.$$

Hence the roots of $\alpha(s)$ lie in \mathbb{S} . The $(n+m-q)$ equations $\tilde{E}k + \tilde{h} = \mathbf{0}$ imply that the $(n+m-q)$ roots of polynomial $\beta(s)$ (see (22)) are placed at $\{-\lambda_1, \dots, -\lambda_{n+m-q}\}$.

Since $x_m = 1$ (see (29)), the corresponding coefficient vector associated with polynomial $x(s)$ would be $x = [x_0 \ x_1 \ \dots \ x_{m-1} \ 1]^T$. Hence the denominator polynomial $x(s)$ of the controller $C(s)$ is a monic polynomial of degree m . The polynomial $y(s)$, on the other hand is of degree not more than m , since there are only $m+1$ entries in vector k corresponding to the polynomial $y(s)$. Hence the resulting controller will either be a proper or strictly proper controller. \square

Remark 3: We have assumed that the plant is strictly proper. Also the resulting controller is proper or strictly proper. Since (see e.g. (L.Qiu and Zhou 2009, Chapter 3, Theorem 3.26)) all the closed loop poles are in the stable region of complex plane, the feedback inter-connection is internally stable.

Note that, the corresponding Sylvester resultant matrix $\mathcal{R}(a, b, 2(m+1))$ for $m < (n-1)$ is a tall matrix and hence there may not exist a k which will satisfy (27) for a specified σ . However, since σ is not completely fixed in our case, (27) may be satisfied for some vector k . According to Theorem 5.2, the controller vector k satisfying the relations (31) and $\tilde{E}k + \tilde{h} = \mathbf{0}$, will guarantee that the pole placement requirements are achieved and (27) is satisfied. The conditions of Theorem 5.2 can be checked by solving the following SDP for increasing values of m .

Problem 5.3 Find $\max_{P,k,\gamma} \gamma$ subject to

$$(i) \quad \tilde{E}k + \tilde{h} = \mathbf{0}$$

$$(ii) \quad \Pi^T(S \otimes P)\Pi - \tilde{F}k\hat{\alpha}_e^T - \hat{\alpha}_e k^T \tilde{F}^T - \tilde{g}\hat{\alpha}_e^T - \hat{\alpha}_e \tilde{g}^T + \gamma I_{q+1} \preceq 0$$

where I_{q+1} is an identity matrix with dimension $q+1$.

To obtain a minimum order controller we have to start with a zero order controller ($m=0$) and check whether the solution γ to Problem 5.3 satisfies $\gamma > 0$. If this condition is not satisfied then we should increase the order of the controller by one and recheck the satisfiability condition. At the stage of $m=(n-1)$ it is guaranteed that the above problem has a feasible solution and hence to obtain the lowest order controller achievable through this method, we need to solve at most n SDPs. The above problem can be solved by using solvers like *SeDuMi* in MATLAB environment (Sturm 2005, Sed 2010).

Remark 4: Often in practical applications, only strictly proper controllers are allowed. The proposed algorithm is applicable in such a situation with some modifications. This can be done by enforcing the m^{th} component (y_m) of the vector y given in (26) to zero. Hence the controller coefficient vector k would be of the dimension $(2m+1)$. For this, the last column of the Sylvester resultant matrix $\mathcal{R}(a, b, 2(m+1))$ need to be deleted from (27). Finally, the design procedure, discussed above, has to be followed after calculating the matrices F, E and vectors g, h as shown in Theorem 5.1. Note that, in this case, the design procedure has to be started with a first order controller ($m=1$).

Remark 5: The proposed approach can also be used to find a (single) reduced order controller for a nonlinear plant operating at multiple operating points. Nonlinear plants, operating at different points are often controlled via linear controllers. For example, in gain scheduled controller designs (Shamma and Athans 1990), the nonlinear plant is required to be linearized at various operating points and (different) linear controllers are designed for each of the resulting linearized plants. While such a design does not necessarily guarantee the stability of the overall nonlinear (switched) system, they are often necessary and extremely successful in practice. A relevant example is aircraft autopilot design, where the linearized airplane model changes frequently depending on atmospheric, altitude and flight mode conditions (Ackermann 1984). As a second example, consider a power system switching to different operating points in response to sudden faults (Pal and Chaudhuri 2005) in the power network. A priori linear models for different fault situations are available and a single controller is supposed to stabilize any of the multiple fault models that the nominal system might change to arbitrarily.

In general, assume that there are w operating points of a nonlinear plant and the linearized models at those points are represented by: $P_l(s) = \frac{b_l(s)}{a_l(s)}$, for $l = 1, 2, \dots, w$ where $a_l(s)$ and $b_l(s)$ are coprime and

are in the form of (18). This leads to the following LMI satisfiability problem:

Problem 5.4 Find $\max_{P,k,\gamma} \gamma$ subject to

$$(i) \quad \tilde{E}_l k + \tilde{h}_l = \mathbf{0}$$

$$(ii) \quad \Pi^T (S \otimes P) \Pi - \tilde{F}_l k \hat{\alpha}_e^T - \hat{\alpha}_e k^T \tilde{F}_l^T - \tilde{g}_l \hat{\alpha}_e^T - \hat{\alpha}_e \tilde{g}_l^T + \gamma I_{q+1} \preceq 0$$

for $l = 1, 2, \dots, w$

Then, a reduced order controller can be computed using the design steps described above. If there is a solution, then the resulting (single) controller should simultaneously satisfy the transient response requirements at all the operating points.

6 Examples

Two numerical examples are presented in this section to illustrate the theory developed above.

6.1 NASA F-8 DFBW aircraft

In this example we consider the linearized model of a NASA F-8 DFBW aircraft (Keel and Bhattacharyya 1990). The lateral dynamics of the aircraft is represented by following matrices:

$$A = \begin{bmatrix} -2.6 & 0.25 & -38 & 0 \\ -0.075 & -0.27 & 4.4 & 0 \\ 0.078 & -0.99 & -0.23 & 0.052 \\ 1 & 0.078 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 17 & 7 \\ 0.82 & -3.2 \\ 0 & 0.046 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The open loop poles of the system are at $-2.3965, -0.0249, -0.3393 \pm 2.6235i$. To obtain a reduced order controller for the above plant, we computed a right co-prime factorization (refer to Structure theorem (Wolovich 1974)) as follows:

$$H(s) = B(s)A(s)^{-1} \quad \text{where}$$

$$A(s) = \begin{bmatrix} s^2 + 2.6062s + 0.7780 & 1.1081s + 5.6244 \\ -0.0533s + 0.8361 & 1.0181s^2 + 0.4938s + 6.5815 \end{bmatrix}$$

$$B(s) = \begin{bmatrix} 0.82s + 0.8113 & -3.2s - 0.2040 \\ 17.0640 & 6.7504 \end{bmatrix}.$$

Note that $A(s)$ is column reduced. According to the design procedure we first consider $v = 1$. Then, the polynomial matrix $W^1(s)$ would be

$$W^1(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ 0 & 1 \\ 0 & s \\ 0 & s^2 \end{bmatrix}.$$

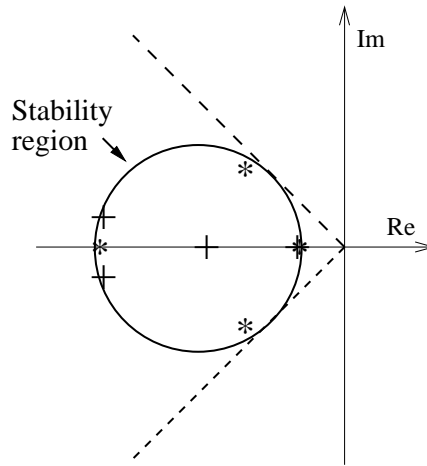


Figure 1. A disc having center at -2.5 and radius 1.7 is chosen as stability region \mathbb{S} . The symbol $+$ shows the position of complex numbers which are used to form $\bar{D}(s)$ and $*$ shows the position of closed loop poles.

Hence, following the procedure discussed in Section 2.3, the eliminant matrix would be as follows:

$$M_1 = \begin{bmatrix} 0.8113 & 0.8200 & 0 & -0.2042 & -3.200 & 0 \\ 17.0640 & 0 & 0 & 6.7504 & 0 & 0 \\ 0.7780 & 2.6062 & 1 & 5.6244 & 1.0181 & 0 \\ 0.8361 & -0.0533 & 0 & 6.5815 & 0.4938 & 1 \end{bmatrix}.$$

Corresponding to this M_1 the matrix L_1 would be as follows:

$$L_1 = \begin{bmatrix} 0.8113 & -0.2042 & 0.8200 & -3.2000 & 0 & 0 \\ 17.0640 & 6.7504 & 0 & 0 & 0 & 0 \\ 0.7780 & 5.6244 & 2.6062 & 1.0181 & 1 & 0 \\ 0.8361 & 6.5815 & -0.0533 & 0.4938 & 0 & 1 \end{bmatrix}.$$

It is required that all the poles of the closed loop system, comprising of plant and a proper controller, should be placed left to a vertical line at -0.2 in the complex plane. In addition, the damping ratio should be greater than or equal to 0.7 . Corresponding to this stability region we have chosen \mathbb{S} as a disc having center at -2.5 and radius 1.7 in the complex plane and hence the elements of S are $s_{11} = 3.36$, $s_{12} = s_{21} = 2.5$ and $s_{22} = 1$. To construct $\bar{D}(s)$ we choose following set of complex numbers: $-2.3965, -0.85$ and $-4 \pm 0.5i$. Note that there is only one open loop pole, that is, -2.3965 is inside \mathbb{S} and hence we have chosen remaining three poles which are close to the boundary of \mathbb{S} , as shown in the Figure 1. Corresponding to the above set of complex numbers the polynomial matrix $\bar{D}(s)$ would be

$$\bar{D}(s) = \begin{bmatrix} s^2 + 3.2465s + 2.0370 & 0 \\ 0 & s^2 + 8s + 16.2500 \end{bmatrix}. \quad (32)$$

Then, by solving Problem 3.3 without considering constraint (ii) we have the following result:

$$X_0 = \begin{bmatrix} 4560.7716 & 2366.8460 \\ -436.5595 & 3144.9283 \end{bmatrix}, \quad Y_0 = \begin{bmatrix} -4625.6015 & 112.7037 \\ -5071.1494 & -139.3457 \end{bmatrix}$$

with closed loop poles are at -4.0362 , $-1.6639 \pm 1.4242i$ and -0.8338 . The locations of the closed loop poles are depicted in Figure 1. Hence we have achieved our objective with a zeroth order controller $C(s) = X_0^{-1}Y_0$. To show the effectiveness of the proposed design procedure for constructing $\bar{D}(s)$ we include the results corresponding to the choice of different $\bar{D}(s)$ in Table 1. Moreover, we obtain the following results by solving Problem 3.3 (with constraint (ii)), corresponding to the choice of $\bar{D}(s)$ as in

Table 1. Results corresponding to the choice of different $\bar{D}(s)$. The diagonal elements of $\bar{D}(s)$ are denoted as $\bar{d}_{11}(s)$ and $\bar{d}_{22}(s)$.

Complex numbers to form $\bar{D}(s)$	Diagonal elements of $\bar{D}(s)$	Resulting X_0 and Y_0	Closed loop poles
$-2.3965, -1,$ $-0.85, -4$	$\bar{d}_{11}(s) = s^2 + 3.3965s + 2.3965$ $\bar{d}_{22}(s) = s^2 + 4.8500s + 3.400$	$X_0 = \begin{bmatrix} 262.1306 & -423.1758 \\ 104.0616 & 494.2231 \end{bmatrix}$ $Y_0 = \begin{bmatrix} 633.7002 & 29.1730 \\ -780.3079 & -25.9751 \end{bmatrix}$	$-3.8975, -1.6131 \pm 1.3061i,$ -0.8821
$-2.3965, -0.9000,$ $-0.8500, -3.5000$	$\bar{d}_{11}(s) = s^2 + 3.2965s + 2.1568$ $\bar{d}_{22}(s) = s^2 + 4.3500s + 2.9750$	$X_0 = \begin{bmatrix} 106.3675 & -169.0821 \\ 24.6396 & 178.2002 \end{bmatrix}$ $Y_0 = \begin{bmatrix} 221.8113 & 13.5346 \\ -266.8398 & -11.9921 \end{bmatrix}$	$-3.1837, -1.7088 \pm 1.2993i,$ -0.9851
$-2.3965, -0.8500,$ $-3 \pm 1.5000i$	$\bar{d}_{11}(s) = s^2 + 3.2465s + 2.0370$ $\bar{d}_{22}(s) = s^2 + 6s + 11.2500$	$X_0 = \begin{bmatrix} 2299.3968 & 910.6293 \\ -544.2107 & 2521.4922 \end{bmatrix}$ $Y_0 = \begin{bmatrix} -1940.1561 & 66.5307 \\ -4057.3566 & -120.9291 \end{bmatrix}$	$-4.0378, -1.6760 \pm 1.4001i,$ -0.8348
$-2.3965, -1,$ $-2.500 \pm 1.600i$	$\bar{d}_{11}(s) = s^2 + 3.3965s + 2.3965$ $\bar{d}_{22}(s) = s^2 + 5s + 8.8100$	$X_0 = \begin{bmatrix} 6672.0534 & 2219.3619 \\ -1881.0828 & 13470.9057 \end{bmatrix}$ $Y_0 = \begin{bmatrix} -4548.3396 & 190.4769 \\ -21438.8574 & -677.5315 \end{bmatrix}$	$-4.0002, -1.6267 \pm 1.4235i,$ -0.8849

(32):

$$X_0 = \begin{bmatrix} 6030.8688 & 3122.8716 \\ -814.2041 & 4054.4896 \end{bmatrix}, \quad Y_0 = \begin{bmatrix} -6143.2737 & 147.7255 \\ -6501.7656 & -191.9227 \end{bmatrix}$$

with closed loop poles are at $-4.0346, -1.6654 \pm 1.4188i$ and -0.8359 . Hence we achieved our objective with a zeroth order controller.

Note that, a zeroth order controller is also computed in Keel and Bhattacharyya (1990) for the above example by solving non-convex optimizations iteratively. On the other hand, we obtained a zeroth order controller by solving a convex optimization corresponding to the choice of $\bar{D}(s)$. The optimization proposed in Keel and Bhattacharyya (1990) uses Sylvester equation for achieving the pole placement requirements. Hence, *a priori* verification of the eigenvalues of pseudo diagonal matrix is required to ensure that they are not close to the open loop poles; as a result one can avoid non-existence of the solution of Sylvester equation. Such necessary actions are not required in the proposed approach. In fact, in the above example we have considered the well damped/stable open loop poles (which are inside \mathcal{S}) to construct $\bar{D}(s)$. We will show in the next example that the proposed algorithm can place a subset of the closed loop poles at specific locations which can not be achieved by the algorithm developed in Keel and Bhattacharyya (1990).

6.2 Four machine two area power system

The performance of the proposed SISO controller design is validated through a case study on a simple power system. A single line diagram of the test system is shown in Figure 2. It comprises of four generators ($G1 - G4$) spread over two geographical areas which are interconnected by two transmission lines. The loads are connected at bus 7 and 9. The details of study system can be found in (Kundur 1994, Pal and Chaudhuri 2005). A thyristor controlled series capacitor (TCSC) (Hingorani and Gyugyi 2000) is installed in the transmission corridor to facilitate power transfer between the two areas. Under normal condition 400 MW power is transferred from *Area 1* to *Area 2* for which the TCSC is set to provide 10% compensation.

Linearized model of the above system about the nominal condition confirms presence of one poorly damped electromechanical mode of oscillation (also known as inter-area oscillation) (Kundur 1994) with about 0.6 Hz frequency. The open loop poles are: $-42.5194, -0.5701 \pm 6.9471i, -0.0467 \pm 3.9352i, -1.9210, 0.0999, -0.7238 \pm 0.7318i$. The objective of this exercise is to improve the damping of this low frequency mode through supplementary control of the TCSC (actuator). The design specification is to achieve a 10 second settling time for this critical mode with a minimum strictly proper controller. This implies shifting the eigenvalues corresponding to the inter-area mode from their open loop position $-0.0467 \pm 3.9352i$ to $-0.4 \pm 3.9352i$ (corresponds to 10 sec. settling time) in closed loop while ensuring that the remaining closed loop poles are restricted to the left of the vertical line at -0.5 in the complex

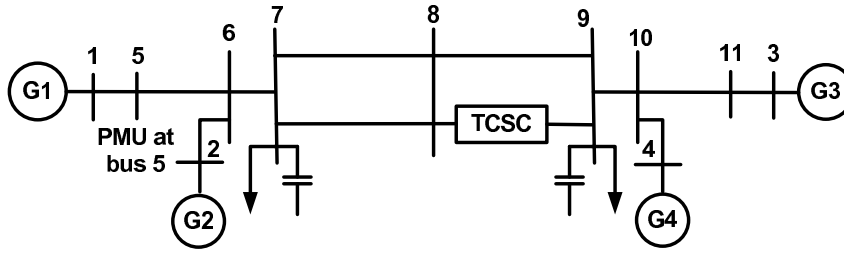


Figure 2. 4 machine 2 area power system with a TCSC. The Phasor Measurement Unit (PMU), which can provide measurements of the states of monitoring buses, is installed at bus 5.

plane. Phase angle difference between the voltages at bus 5 and 11 was chosen as the feedback signal due to its highest modal observability. A 9^{th} order linearized equivalent model is considered for the power system and a strictly proper reduced order controller is designed following the proposed approach.

6.2.1 Controller Design

As discussed above, the stability region for the free poles would be the closed left half of a vertical line at -0.5 in the complex plane. Hence the stability region (2) will take the following form:

$$\mathbb{S} = \left\{ s \in \mathbb{C} : [1 \ s^*] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0 \right\} \quad (33)$$

According to the design steps, we first try with 1^{st} order and subsequently 2^{nd} order controller to achieve our objectives. However, no feasible solutions exist for Problem 5.3 at these stages. Hence the next step is to try with a 3^{rd} order controller and it is observed that then Problem 5.3 does have a feasible solution.

Third order controller : The order of the plant $P(s)$ is $n = 9$. The order of the controller $C(s)$ is 3. Hence, the number of closed loop poles is twelve. Among them two poles (corresponding to the inter-area mode) are critical and hence already specified. The remaining 10 poles can take any positions in the stability region defined in (33). To form the central polynomial $\hat{\alpha}(s)$, the following poles are chosen inside the stability region \mathbb{S} :

$$\{-42.5194, -0.5701 \pm 6.9471i, -1.9210, -0.7238 \pm 0.7318i, -0.55, -0.8, -0.6 \pm 1i\}$$

Note that the open loop poles $-42.5194, -0.5701 \pm 6.9471i, -1.9210, -0.7238 \pm 0.7318i$ are inside the stability region \mathbb{S} and hence we consider them to form the central polynomial. The remaining poles to construct $\hat{\alpha}(s)$ are chosen close to the boundary of \mathbb{S} .

The matrices F , E and vectors g , h are calculated following Section 5. Solving Problem 5.3, the resulting controller coefficient vector turns out to be

$$k = [39.1532 \ 32.0201 \ 13.7534 \ 1 \ 433.1596 \ 174.5694 \ 20.4935]^T,$$

and hence the corresponding 3^{rd} order controller would be:

$$C(s) = \frac{20.4945s^2 + 174.5694s + 433.1596}{s^3 + 13.7534s^2 + 32.0201s + 39.1532}.$$

The closed loop poles are given in Table 2. It is clear that the poles corresponding to the inter-area mode are placed at the desired locations and their settling time should be less than 10 second. Furthermore, all the free poles have assumed positions in \mathbb{S} as defined in (33). Hence all the requirements on closed loop poles are achieved with a 3^{rd} order strictly proper controller.

The damping ratio (ζ), frequency of oscillation (f) and settling time (t_s) of inter-area modes for the open loop and closed loop plants are shown in Table 3.

Table 2. Pole Locations Table

Closed loop poles	ζ	t_s in sec.
$-42.8375, -10.9750, -1.5865, -1.4234$	1	
$-0.5637 \pm 6.9493i$	0.08	7.09
$-0.4000 \pm 3.9352i$	0.11	10
$-0.5332 \pm 0.8012i$	0.55	7.50
$-0.5794 \pm 0.0715i$	0.99	6.90

Table 3. Comparison Table

Condition	Inter-area Modes	f in Hz	ζ	t_s in sec.
Open Loop	$-0.0467 \pm 3.9352i$	0.62	0.01	86.65
Closed Loop	$-0.4000 \pm 3.9352i$	0.62	0.11	10

7 Conclusion

In this article we propose a design framework to obtain a reduced order output feedback controller for linear systems while guaranteeing that the closed loop poles are placed within some pre-specified region in the complex plane. In addition the proposed method can achieve partial pole placement for SISO systems while optimising the controller order. This combination of objectives, though clearly important, has been rarely treated in the literature. Moreover, after suitable approximations, the proposed sub-optimal controller order minimisation algorithm is convex, corresponding to a fixed choice of stable polynomial matrix and solvable using standard semidefinite programming tools. Some of the conservativeness in the methodology developed here stems from (i) the inner approximation method used for convexifying the stability region in the coefficient space, and (ii) the use of constraint (ii) of Theorem 3.2 to convexify the requirement of row-reduced $X(s)$. Furthermore, the optimal solution of Problem 3.3 with constraint (ii) produces a controller having order equal to $m(v-1)$, whereas without imposing it the controller order could be less than or equal to $m(v-1)$. Hence, we propose to solve Problem 3.3, first, without considering constraint (ii); and we have experienced through the numerical examples that the solution of Problem 3.3 without constraint (ii) usually results in a row-reduced polynomial matrix $X(s)$. While these problems are currently being investigated, the proposed algorithm seems to perform well in numerical examples.

References

- “SeDuMi 1.3,” <http://sedumi.ie.lehigh.edu> (2010).
- Ackermann, J. (1980), “Parameter space design of robust control systems,” *IEEE Transactions on Automatic Control*, 25, 1058–1072.
- Ackermann, J. (1984), “Robustness against sensor failures,” *Automatica*, 20, 211–215.
- Antsaklis, P.J., and Michel, A.N., *Linear Systems*, Birkhauser Boston (2006).
- Bernstein, D. (1992), “Some open problems in matrix theory arising in linear systems and control,” *Linear Algebra and its Applications*, pp. 409–432.
- Datta, S., and Chakraborty, D. (2013), “Minimal state measurements for regional pole placement,” in *Proceedings of the IEEE American Control Conference*, Washington, DC, pp. 2397–2402.
- Datta, S., and Chakraborty, D. (2014), “Feedback Norm Minimization with Regional Pole Placement,” *International Journal of Control*,, accepted for publication.
- Datta, S., Chakraborty, D., and Belur, M. (2012b), “Low Order Controller with Regional Pole Placement,” in *Proceedings of the IEEE American Control Conference*.
- Datta, S., Chakraborty, D., and Chaudhuri, B. (2012a), “Partial Pole Placement with Controller Optimization,” *IEEE Transactions on Automatic Control*, 57, 1051–1056.
- Hammer, J. (1983), “Pole assignment and minimal feedback design,” *International Journal of Control*, 37, 63–88.
- Han, J.H., Oliveira, M.C.D., and Skelton, R.E. (2006), “Linearization algorithm for a reduced order H_∞ control design of an active suspension system,” *European Journal of Control*, 12, 205–219.
- Henrion, D., Arzelier, D., and Peaucelle, D. (2003), “Positive polynomial matrices and improved LMI

- robustness conditions,” *Automatica*, 39, 1479–1485.
- Henrion, D., Sebek, M., and Kucera, V. (2003), “Positive polynomials and robust stabilization with fixed-order controllers,” *IEEE Transaction on Automatic Control*, 48, 1178–1186.
- Hingorani, N.G., and Gyugyi, L., *Understanding FACTS: concepts and technology of flexible AC transmission systems*, New York: IEEE Press (2000).
- Horn, R., and Johnson, C., *Matrix Analysis*, Cambridge University Press (1985).
- Kailath, T., *Linear Systems*, Englewood Cliffs, Prentice-Hall (1980).
- Karimi, A., Khatibi, H., and Longchamp, R. (2007), “Robust control of polytopic systems by convex optimization,” *Automatica*, 43, 1395–1402.
- Keel, L., and Bhattacharyya, S.P. (1990), “State space design of low order stabilizers,” *IEEE Transactions on Automatic Control*, 35, 182–186.
- Khatibi, H., Karimi, A., and Longchamp, R. (2008), “Fixed-Order Controller Design for Polytopic Systems Using LMIs,” *IEEE Transactions on Automatic Control*, 53, 428–434.
- Kundur, P., *Power System Stability and Control*, McGraw-Hill : New York, London, The EPRI power system engineering series (1994).
- L.Qiu., and Zhou, K., *Introduction to Feedback Control*, Upper Saddle River, New Jersey: Prentice Hall (2009).
- Luenberger, D. (1964), “Observing the State of a Linear System,” *Military Electronics, IEEE Transactions on*, 8, 74–80.
- Mesbahi, M. (1998), “On the rank minimization problem and its control applications,” *Systems and Control Letters*, 33, 31–36.
- Mesbahi, M., and Papavassilopoulos, G. (1997), “On the rank minimization problem over a positive semi-definite linear matrix inequality,” *IEEE Transactions on Automatic Control*, 42, 239–243.
- Obinata, G., and Anderson, B.D.O., *Model Reduction for Control System Design*, Springer-Verlag London (2001).
- Pal, B., and Chaudhuri, B., *Robust Control in Power systems*, Springer-New York, Power Electronics and Power systems (2005).
- Shamma, J.S., and Athans, M. (1990), “Analysis of gain scheduled control for nonlinear plants,” *IEEE Transaction on Automatic Control*, 35, 898–907.
- Sturm, J.F. (2005), “Using *SeDuMi* 1.02, a Matlab Toolbox for optimization over symmetric cones,” *Optimization Methods Software*, cs.SC, 625–653.
- Wang, S., and Chow, J. (2000), “Low-order controller design for SISO systems using coprime factors and LMI,” *IEEE Transactions on Automatic Control*, 45, 1166–1169.
- Wellstead, K. (1991), “Robust Polynomial Controller Design,” Brunel University, U.K.
- Wolovich, W., *Linear Multivariable Systems*, Springer Verlag NY, Berlin (1974).
- Yang, F., Gani, M., and Henrion, D. (2007), “Fixed-Order Robust H_∞ Controller Design With Regional Pole Assignment,” *IEEE Transaction on Automatic Control*, 52, 1959–1963.