

RESEARCH ARTICLE

Feedback Norm Minimization with Regional Pole Placement

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This article proposes a convex algorithm for minimizing an upper bound of the state feedback gain matrix norm with regional pole placement for LTI multi-input systems. The inherent non-convexity in this optimization is resolved by a combination of two separate approaches: (i) an inner convex approximation of the polynomial matrix stability region due to Henrion and (ii) a novel convex parameterization of column reduced matrix fraction system representations. Using a sequence of approximations enabled by the above two methods, it is shown that the constraints on closed loop poles (both pre-specified exact locations and regional placement) define linear matrix inequalities. Finally, the effectiveness of the proposed algorithm is compared with similar pole placement algorithms through numerical examples.

Keywords: Linear systems, LMIs, Convex optimizations, Pole placement

1 Introduction

Pole placement algorithms for state feedback controllers are well studied in linear control and a variety of methods exist for solving this problem. Given n self conjugate complex numbers $(\lambda_1, \dots, \lambda_n)$ and a controllable linear time invariant system $\dot{x} = Ax + Bu$, exact pole placement algorithms try to compute a state feedback matrix F such that eigenvalues of $(A - BF)$ coincide with $(\lambda_1, \dots, \lambda_n)$. For multi-input systems such an F is non-unique (Wonham 1974) and several algorithms (e.g. see (Kautsky et al. 1985, Chu 2001, Miminis and Paige 1982, Mehrmann and XU 1997, Tam and Lam 1997, Varga 2000a,b) and references therein) optimize various aspects of the F matrix and associated quantities based on the degrees of freedom available in F . It was shown in (Mehrmann and XU 1997) that relevant quantities for such optimization formulations are the state feedback matrix norm (Keel et al. 1985, Wang and Chow 2000, Kouvaritakis and Cameron 1980, Varga 2000a,b), the condition number of the associated eigenvector matrix (Kautsky et al. 1985, Chu 2001, Miminis and Paige 1982, Mehrmann and XU 1997, Tits and Yang 1996, Tam and Lam 1997, Varga 2000a,b, Rami et al. 2009) and the distance to uncontrollability (Mehrmann and XU 1997). These optimization problems are rarely convex and have varying numerical properties (Rami et al. 2009). A second variety of algorithms solve the regional pole placement problem: i.e. places the eigenvalues of $(A - BF)$ somewhere within a pre-specified region in the complex plane. The degrees of freedom (from the non-uniqueness of F and from the regional placement requirement) is utilized to minimize the H_∞ and/or H_2 norm of the closed loop transfer function (Chilali et al. 1999, Goncalves et al. 2004, Chilali and Gahinet 1996), while guaranteeing regional pole placement. These algorithms however have no direct control over the feedback matrix properties. In this article, we propose a convex method to minimize an upper bound of the feedback matrix norm, while ensuring regional pole placement. To the best of our knowledge, this combination of objectives, resulting especially in a convex optimization, is not available in the literature.

Very often in practice, the control system performance specifications mention time domain characteristics and not exact desired pole locations. Hence the regional pole placement paradigm is relevant in

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most practical situations. On the other hand, every actuator can provide only a limited control effort and the cost of the actuator grows quickly with the cost of the maximum effort that it is required to produce. Since the maximum control effort is directly proportional to the norm of the feedback gain matrix, the minimum feedback gain satisfying the regional placement of poles is of interest. A method for addressing this problem for single input systems was proposed in (Datta et al. 2010, 2011, 2012). This article generalizes those results for multi-input systems. The method proposed here, with minor modifications, can also handle exact pole placement requirements.

Most exact pole placement methods aiming to optimize aspects of the feedback matrix F use the Sylvester parameterization (Varga 2000b, Keel et al. 1985):

$$AQ - QA_F - BFQ = 0 \quad (1)$$

where Q is a nonsingular transformation and the matrix A_F contains the desired closed loop eigenvalues. Then, any F and Q satisfying (1) solves the pole placement problem. However, (1) often leads to a *non-convex* optimization for the norm of F (Varga 2000a,b). In this article, we propose a different approach to **parameterize** the non-uniqueness of F . It is well known (Kailath 1980) that a pair (A, B) is associated with infinitely many matrix fraction descriptions (MFDs) of the form $\tilde{R}(s)\tilde{P}^{-1}(s)$, with each of these representations being related by unimodular polynomial matrix transformations (say $U(s)$). Hence we first parameterize the closed loop MFD as follows: $(sI - A + BF)^{-1}B = \tilde{R}(s)U(s) \left[\{\tilde{P}(s) + F\tilde{R}(s)\}U(s) \right]^{-1}$.

Then any F and $U(s)$ pair which makes the zeros of $|\{\tilde{P}(s) + F\tilde{R}(s)\}U(s)|$ (where $|\cdot|$ denotes determinant) satisfy the pole placement constraints are candidate solutions. In conjunction with a non-linear transformation of the coefficients of $P_F(s) := \{\tilde{P}(s) + F\tilde{R}(s)\}U(s)$ and an inner approximation of polynomial matrix stability region (Henrion et al. 2003), this parameterization of $P_F(s)$ is used to convexify the regional pole placement constraints. The resulting norm minimization problem is then reduced to an linear matrix inequality (LMI) constrained optimization solvable by existing LMI solvers.

Rest of the paper is organized as follows. In Section 2 the problem is formulated and some known results are reviewed. Section 3 describes a characterization of unimodular matrices associated with column reduced polynomial matrices. Finally, the problem is formulated as an LMI optimization in Section 4. A comparison between the proposed algorithm and some existing norm minimizing algorithms is discussed in Section 5 with examples.

2 Problem Formulation and Preliminaries

2.1 Problem Formulation

Consider a continuous time LTI multi-input system, represented by the following state space equation

$$\dot{x} = Ax + Bu \quad (2)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ is full rank. Assume that the pair (A, B) is completely controllable. Then, with a linear state feedback control law: $u = -Fx$ where $F \in \mathbb{R}^{m \times n}$, it is possible to modify *all* the pole locations of the closed loop system

$$\dot{x} = (A - BF)x \quad (3)$$

in the complex plane (Kailath 1980, Wolovich 1974).

Next, define a stability region \mathbb{S} in the complex plane as follows:

$$\mathbb{S} = \{s \in \mathbb{C} : s_{11} + s_{12}(s + s^*) + s_{22}ss^* < 0\} \quad (4)$$

where s^* denotes the complex conjugate of s . It is shown in (Henrion et al. 2003) that the region \mathbb{S} can

be used to represent some common stability regions in the complex plane (like arbitrary half planes and discs). Then, the regional pole placement problem, discussed in the introduction, can be precisely posed as follows:

Problem 2.1 Find $\inf \|F\|_F$ such that the eigenvalues of $(A - BF)$ are placed anywhere in \mathbb{S} .

We briefly overview some definitions related to polynomial matrices and MFDs in the following section.

2.2 Polynomial Matrices

Let us denote $\mathbb{R}[s]$ and $\mathbb{R}^{m \times m}[s]$ as the set of all polynomials and $(m \times m)$ polynomial matrices respectively. Consider a polynomial matrix $P(s) \in \mathbb{R}^{m \times m}[s]$ with its entries $p_{ij}(s) \in \mathbb{R}[s]$ for $i, j = 1, 2, \dots, m$. Let r be the highest degree occurring among the degrees of the polynomial entries $p_{ij}(s)$. Then the polynomial matrix $P(s)$ can be represented as

$$P(s) = P_r s^r + P_{r-1} s^{r-1} + \dots + P_1 s + P_0 \quad (5)$$

where $P_k \in \mathbb{R}^{m \times m}$ for $k = 0, 1, 2, \dots, r$ are the coefficient matrices of $P(s)$. Henceforth we will denote P_k as the k^{th} coefficient of $P(s)$ and r as the *degree* of the polynomial matrix (Wolovich 1974). Likewise the maximum degree occurring among the degrees of all elements in the j^{th} column, $p_j(s)$ of polynomial matrix $P(s)$, is referred to *column degree* of $p_j(s)$ and denoted as $\delta(p_j)$. A polynomial matrix $P(s) \in \mathbb{R}^{m \times m}[s]$ is said to be \mathbb{S} -stable if all the *zeros* of $P(s)$ (i.e. roots of $|P(s)| = 0$, where $|\bullet|$ denotes the determinant) belong to some stability region \mathbb{S} in the complex plane.

Let us assume that the column degrees of a polynomial matrix $P(s) \in \mathbb{R}^{m \times m}[s]$ are $\delta(p_j) = d_j$ for $j = 1, 2, \dots, m$. Then $P(s)$ can always be written as $P(s) = P_h D(s) + P_l(s)$ where $D(s) = \text{diag}\{s^{d_1}; s^{d_2}; \dots; s^{d_m}\}$, (where $\text{diag}\{\bullet\}$ denotes the diagonal matrix) $P_h \in \mathbb{R}^{m \times m}$ is the *highest column degree coefficient matrix* of $P(s)$ and $P_l(s)$ is the polynomial matrix consisting of remaining lower degree terms of $P(s)$. We say that $P(s)$ is *column reduced* if $|P_h| \neq 0$ (Kailath 1980, Wolovich 1974).

For a given controllable (A, B) pair we can write $(sI - A)^{-1}B = \tilde{R}(s)\tilde{P}^{-1}(s)$ where $\tilde{R}(s)$ and $\tilde{P}(s)$ are polynomial matrices (Wolovich 1974, Kailath 1980). However, this matrix factorization is not unique since by defining

$$R(s) := \tilde{R}(s)U(s) \text{ and } P(s) := \tilde{P}(s)U(s) \quad (6)$$

we can write $(sI - A)^{-1}B = R(s)P^{-1}(s)$ where $U(s)$ is a unimodular polynomial matrix. Similarly, a matrix factorization associated with the closed loop system would be

$$(sI - A + BF)^{-1}B = R(s)P_F^{-1}(s) \quad (7)$$

where $P_F(s) = P(s) + FR(s)$ (see (Wolovich 1974, Kucera 1981) and the references therein). By equating the denominator polynomials in (7) we can write

$$|(sI - A + BF)| = |P_F(s)| = |(P(s) + FR(s))|$$

which says that the closed loop poles of (3) are captured in the polynomial matrix $P_F(s)$ and are equal to the zeros of the polynomial $|P_F(s)|$.

2.3 Companion Form and Structure Theorem

We briefly review the controllable companion form and some associated MFDs. Recall that the multi-variable controllable companion structure (\hat{A}, \hat{B}) corresponding to (A, B) can be obtained by a similarity

transformation of the form $\hat{A} = Q^{-1}AQ$ and $\hat{B} = Q^{-1}B$, where $Q \in \mathbb{R}^{n \times n}$ is invertible, and (\hat{A}, \hat{B}) is of the form :

$$\hat{A} = \left[\begin{array}{c|ccc|ccc|ccc} 0 & 1 & 0 & \cdots & 0 & & & & & & & \\ 0 & 0 & 1 & \cdots & 0 & & & & & & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \circ & & \cdots & & \circ & \\ 0 & 0 & 0 & \cdots & 1 & & & & & & & \\ \times & \times & \times & \cdots & \times & \times & \times & \times & \cdots & \times & & \\ \hline & & & & & 0 & 1 & 0 & \cdots & 0 & & \\ & & & & & 0 & 0 & 1 & \cdots & 0 & & \\ & & & & & \vdots & \vdots & \vdots & \ddots & \vdots & & \\ & & & & & 0 & 0 & 0 & \cdots & 1 & & \\ \times & \times & \times & \cdots & \times & \times & \times & \times & \cdots & \times & & \\ \hline & & & & & \vdots & & & & & & \\ & & & & & \vdots & & & & & & \\ & & & & & \vdots & & & & & & \\ & & & & & & & & & & 0 & 1 & 0 & \cdots & 0 \\ & & & & & & & & & & 0 & 0 & 1 & \cdots & 0 \\ & & & & & & & & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & & & & & & & 0 & 0 & 0 & \cdots & 1 \end{array} \right] ; \hat{B} = \left[\begin{array}{cccc} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & \times & \times & \cdots & \times \\ \hline 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \times & \cdots & \times \\ \hline \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{array} \right]$$

Assuming d_j 's for $j = 1, 2, \dots, m$ as *controllability indices* (Wolovich 1974) of the plant (2); let us define $\mu_k := \sum_{j=1}^k d_j$ for $k = 1, 2, \dots, m$. Note that $\mu_m = d_1 + d_2 + \dots + d_m = n$. Now construct $\hat{A}_m \in \mathbb{R}^{m \times n}$ and $\hat{B}_m \in \mathbb{R}^{m \times m}$ by taking the m ordered μ_k^{th} rows of \hat{A} and \hat{B} . It is clear that \hat{B}_m is a non-singular matrix. Then, the input to state transfer function matrix $(sI - A)^{-1}B = Q^{-1}(sI - \hat{A})^{-1}\hat{B}$. Next, define the following two polynomial matrices:

$$W(s) := \text{diag}\{1, s, \dots, s^{d_1-1}; 1, s, \dots, s^{d_2-1}; \dots; 1, s, \dots, s^{d_m-1}\} \quad (8)$$

and $D(s) := \text{diag}\{s^{d_1}; s^{d_2}; \dots; s^{d_m}\}$. Then, according to the structure theorem discussed in (Wolovich 1974), we can write

$$(sI - A)^{-1}B = Q^{-1}(sI - \hat{A})^{-1}\hat{B} = \tilde{R}(s)\tilde{P}^{-1}(s) \quad (9)$$

where

$$\tilde{R}(s) = Q^{-1}W(s), \quad \tilde{P}(s) = \left[\hat{B}_m^{-1} \left(D(s) - \hat{A}_m W(s) \right) \right], \quad (10)$$

the polynomial matrix $\tilde{P}(s)$ is column reduced and the column degree of $P(s)$ is $\delta(\tilde{p}_j) = d_j$ for $j = 1, 2, \dots, m$.

3 Parameterization of MFDs

Recall, (Varga 2000b, Keel et al. 1985) the Sylvester equation:

$$AQ - QA_F - BFQ = 0 \quad (11)$$

where Q is a nonsingular transformation matrix and A_F is a design matrix whose eigenvalues are chosen to be the desired closed loop eigenvalues. Then, any F and Q satisfying (11) solves the pole placement problem. However, it is shown in (Varga 2000a,b) that the parameterization (11) leads to a *non-convex* optimization while computing a minimum norm feedback gain matrix F . We show that this issue can be overcome by using the following MFD based parameterization instead of the Sylvester's equation.

Recall (7), where any arbitrary right MFD representing the closed loop system (3) can be represented as $R(s)P_F^{-1}(s)$ with $P_F(s)$ parameterized in terms of all possible unimodular matrices $U(s)$:

$$P_F(s) = \left[\tilde{P}(s) + F\tilde{R}(s) \right] U(s) \tag{12}$$

We have already seen that the polynomial matrix $\tilde{P}(s)$ in (10) is column reduced and $\delta(\tilde{p}_j) = d_j$. Now, if we assume $P(s) = \tilde{P}(s)U(s)$ to be column reduced and column degrees of $\tilde{P}(s)$ and $P(s)$ are arranged in order, then it is known (Kailath 1980, Section 6.3) that

$$\delta(p_j) = \delta(\tilde{p}_j) = d_j \text{ for } j = 1, 2, \dots, m. \tag{13}$$

This leads to the following result:

Lemma 3.1: *Let us assume that the controllability indices of system (2) are arranged in the following order: $d_1 > d_2 > \dots > d_m$. Then, $P(s) = \tilde{P}(s)U(s)$ (with $U(s) \in \mathbb{R}^{m \times m}[s]$) is column reduced and the column degrees are arranged in descending order if and only if $U(s)$ is lower triangular with non-zero scalars in the diagonal.*

Proof Only if part: Assume that $P(s)$ is column reduced and column degrees of $P(s)$ are arranged in descending order. Denote $u_j(s)$ for $j = 1, \dots, m$ as the j^{th} column of $U(s)$ and $u_{ij}(s)$ as i^{th} element of $u_j(s)$. Since $P(s) = \tilde{P}(s)U(s)$, it is clear that

$$p_j(s) = \tilde{P}(s)u_j(s)$$

where $p_j(s)$ is the j^{th} column of $P(s)$. Then, the degree of j^{th} column, for $j = 1, 2, \dots, m$, must satisfy:

$$\delta(p_j) = \max_i [\delta(u_{ij}) + d_i] \text{ for } i = 1, 2, \dots, m$$

where $\delta(u_{ij})$ is the degree of $u_{ij}(s)$ (predictable degree property of column reduced polynomial matrices (Kailath 1980, Theorem 6.3-13)). Then, using (13) we have

$$d_j = \max_i [\delta(u_{ij}) + d_i] \text{ for } i = 1, 2, \dots, m \tag{14}$$

According to the relation (14), the elements $u_{ij}(s)$ would be in the following form:

$$u_{ij}(s) = \begin{cases} u_{ij}^{d_j-d_i} s^{d_j-d_i} + \text{LDTR}, & \text{for } j \leq i, i = 1, 2, \dots, m \\ 0, & \text{otherwise.} \end{cases}$$

where LDTR stands for lower degree terms. Hence

$$U(s) = \begin{bmatrix} u_{11}^0 & 0 & 0 & \dots & 0 \\ u_{21}(s) & u_{22}^0 & 0 & \dots & 0 \\ u_{31}(s) & u_{32}(s) & u_{33}^0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{m1}(s) & u_{m2}(s) & u_{m3}(s) & \dots & u_{mm}^0 \end{bmatrix} \tag{15}$$

and $U(s)$ would be unimodular if and only if u_{ii}^0 's for $i = 1, 2, \dots, m$ are not zero.

If part: Assume that $U(s)$ is in the form of (15). It is shown in (Kailath 1980, Section 6.3) that a polynomial matrix $P(s)$ will be column reduced if

$$\text{deg}(|P(s)|) = \sum_{j=1}^m \delta(p_j) \tag{16}$$

where $\text{deg}(\bullet)$ denotes the degree of a polynomial. It is clear that $\text{deg}(|P(s)|) = \text{deg}(|\tilde{P}(s)U(s)|) = \text{deg}(|\tilde{P}(s)|) = n$. Furthermore, according to the construction of $U(s)$ we have $\delta(p_j) = d_j$. However $\sum_{j=1}^m d_j = n$. Hence $P(s)$ satisfies (16) and hence it is column reduced and column degrees are in descending order. \square

In Lemma 3.1 we assumed that the controllability indices are satisfying strict inequality. The general case is considered next.

Theorem 3.2: *Let us assume that a subset of the controllability indices of plant (2) are arranged in the following order:*

$$d_k > d_{k+1} > \dots > d_p > d_{p+1} = d_{p+2} = \dots = d_t \tag{17}$$

where $1 \leq k \leq p$ and $p+1 \leq t \leq m$. Then, $P(s) = \tilde{P}(s)U(s)$ (with $U(s) \in \mathbb{R}^{m \times m}[s]$) is column reduced with corresponding column degrees are arranged according to (17) if and only if $U(s)$ is lower block triangular and the corresponding sub-blocks would be of the form

$$U_{kt}(s) = \begin{bmatrix} p-k+1 & t-p \\ U_{11}(s) & \bigcirc \\ * & U_{22} \end{bmatrix} \begin{matrix} p-k+1 \\ t-p \end{matrix} \tag{18}$$

where the matrix $U_{11}(s)$ is a lower triangular polynomial matrix with non-zero scalars in the diagonal and U_{22} is a real matrix.

Proof Only if part: Let us assume that $P(s)$ is column reduced and column degrees are ordered according to (17). Then, following the proof of Lemma 3.1, the elements of $U(s)$, for $j = k, \dots, p$, would be:

$$u_{ij}(s) = \begin{cases} u_{ij}^{d_j-d_i} s^{d_j-d_i} + LDTR, & \text{for } j \leq i, i = k, k+1, \dots, t \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, to satisfy (14), the elements of $U(s)$, for $j = p+1, p+2, \dots, t$ would be

$$u_{ij}(s) = \begin{cases} 0, & \text{for } i = 1, 2, \dots, p \\ u_{ij}^0 \in \mathbb{R}, & \text{for } i = p+1, p+2, \dots, t. \end{cases}$$

Hence the resulting $U_{kt}(s)$ is in the form of (18), and $U_{kt}(s)$ would be a unimodular polynomial matrix if and only if U_{22} is nonsingular and all the diagonal elements of $U_{11}(s)$ are non-zero.

If part: The proof is analogous to the proof described in *if* part of Lemma 3.1. \square

The degree (defined in Section 2.2) of the resulting unimodular matrix $U(s)$ is computed next.

Corollary 3.3: *Let us assume that the controllability indices of system (2) are arranged as follows: $d_1 \geq d_2 \geq \dots \geq d_m$. Then, the degree of unimodular matrix $U(s)$ is $v := d_1 - d_m$.*

Proof According to the construction of $U(s)$ it is clear that for a fixed $j = 1, 2, \dots, m$ the degree of polynomials $u_{ij}(s)$ will satisfy

$$\text{deg}(u_{mj}) \geq \text{deg}(u_{ij}), \text{ for } i = 1, 2, \dots, m-1 \tag{19}$$

and hence the degree of column $u_j(s)$ is equal to the degree of the polynomial $u_{mj}(s)$. Furthermore, the last row polynomials are $u_{mj}(s) = u_{mj}^{d_j-d_m} s^{d_j-d_m} + LDTR$. According to the arrangement of controllability indices, we have $(d_j - d_m) \geq (d_{j+1} - d_m)$ for $j = 1, 2, \dots, m-1$. Hence the degree of $U(s)$ is v . \square

Example 3.4 Let us consider a plant whose controllability indices are as follows: $d_1 = 5$, $d_2 = 3$ and $d_3 = d_4 = 2$. Then, according to the above construction the unimodular matrix would be:

$$U(s) = \left[\begin{array}{cc|cc} & u_{11}^0 & 0 & 0 & 0 & 0 \\ & u_{21}^2 s^2 + u_{21}^1 s + u_{21}^0 & u_{22}^0 & 0 & 0 & 0 \\ \hline u_{31}^3 s^3 + u_{31}^2 s^2 + u_{31}^1 s + u_{31}^0 & u_{32}^1 s + u_{32}^0 & u_{33}^0 & u_{34}^0 & & \\ u_{41}^3 s^3 + u_{41}^2 s^2 + u_{41}^1 s + u_{41}^0 & u_{42}^1 s + u_{42}^0 & u_{43}^0 & u_{44}^0 & & \end{array} \right]$$

and its degree is $d_1 - d_4 = 3$.

Following Corollary 3.3, the resulting unimodular matrix $U(s)$ can be written as:

$$U(s) = U_0 + U_1 s + \dots + U_{v-1} s^{v-1} + U_v s^v.$$

Furthermore, let d_1 be the largest among all d_j 's for $j = 1, 2, \dots, m$, then the degree of polynomial matrices $\tilde{P}(s)$ and $P(s)$ would be d_1 . By stacking the coefficients of $\tilde{P}(s)$ and $P(s)$, let us define

$$\mathbf{P} := [P_0 \ P_1 \ P_2 \ \dots \ P_{d_1-1} \ P_{d_1}] \in \mathbb{R}^{m \times (d_1+1)m}; \quad \tilde{\mathbf{P}} := [\tilde{P}_0 \ \tilde{P}_1 \ \tilde{P}_2 \ \dots \ \tilde{P}_{d_1-1} \ \tilde{P}_{d_1}] \in \mathbb{R}^{m \times (d_1+1)m}.$$

Since $P(s) = \tilde{P}(s)U(s)$, it can be shown that

$$\mathbf{P} = \tilde{\mathbf{P}}\mathbf{U} \tag{20}$$

where

$$\mathbf{U} = \begin{bmatrix} U_0 & U_1 & \dots & U_v & \bigcirc \\ & U_0 & \ddots & \ddots & \\ & & \ddots & \ddots & U_v \\ & & & \ddots & \vdots \\ \bigcirc & & & U_0 & U_1 \\ & & & & U_0 \end{bmatrix} \in \mathbb{R}^{(d_1+1)m \times (d_1+1)m} \tag{21}$$

In the following section we utilize the parameterization (12) and (20) for regional pole placement as well as fixed pole placement problems. By exploiting the freedom associated with the choice of unimodular matrix $U(s)$, constructed by the procedure discussed above, we develop methodologies to solve Problem 2.1.

4 Optimization Formulation for Minimum Norm Feedback Gain Matrix

Recall that in Problem 2.1 we are interested in finding a minimum norm feedback gain matrix F such that all the closed loop poles belong to some pre-defined stability region \mathbb{S} . For this purpose let us define the following set

$$\mathcal{N}_{\mathbb{S}} := \{P_F(s) \in \mathbb{R}^{m \times m}[s] : \text{all zeros of } P_F(s) \in \mathbb{S}\}. \tag{22}$$

Then Problem 2.1 can be rewritten as: find $\inf \|F\|_F$ such that the polynomial matrix $P_F(s) \in \mathcal{N}_{\mathbb{S}}$. However, the set $\mathcal{N}_{\mathbb{S}}$ is not a convex set (Henrion et al. 2003) and hence optimization over this set becomes non-convex. To convexify the above problem we will use a result by (Henrion et al. 2003) to compute a convex inner approximated set of $\mathcal{N}_{\mathbb{S}}$ in the following section. Before that let us determine the degree of the polynomial matrix $P_F(s)$ through the following result.

Corollary 4.1: *Assume that the controllability indices of system (2) are arranged as follows: $d_1 \geq d_2 \geq \dots \geq d_m$. Then, the degree of the polynomial matrix $P_F(s)$ would be d_1 .*

Proof We have already shown in (12) that $P_F(s) = [P(s) + FR(s)] = [P(s) + FQ^{-1}W(s)U(s)]$. Assume that the unimodular matrix $U(s)$ is constructed according to Theorem 3.2. Let us define a polynomial matrix $K(s) := W(s)U(s)$. Denote $k_j(s)$ as the j^{th} column of $K(s)$ and $w_i^T(s)$ as the i^{th} row of the matrix $W(s)$. Then, it is clear that for a fixed column j we have

$$\delta(k_j) = \max_i [deg(w_i^T(s)u_j(s))], \text{ for } i = 1, 2, \dots, n.$$

Then, it directly follows from the definition of $W(s)$ (see (8)) and construction of $U(s)$ that $\delta(k_j) = d_j - 1$. Hence the degree of j^{th} column of the polynomial matrix $FR(s) = FQ^{-1}K(s)$ is equal to $d_j - 1$. We have already shown that $\delta(p_j) = d_j$ where $p_j(s)$'s are the columns of $P(s) = \tilde{P}(s)U(s)$. Hence, the degree of the j^{th} column of $P_F(s) = P(s) + FR(s)$ would be d_j . Since d_1 is the largest among all d_j 's, the degree of the polynomial matrix $P_F(s)$ would be d_1 . \square

In the rest of the paper we assume that d_1 is the largest among all d_j 's for $j = 1, 2, \dots, m$.

4.1 LMI Stability Region

Let us define following two matrices corresponding to the polynomial matrices $P_F(s)$ and an arbitrary but fixed $\bar{P}_F(s)$ (of degree d_1) respectively:

$$\mathbf{P}_F := [P_{F_0} \ P_{F_1} \ \dots \ P_{F_{d_1-1}} \ P_{F_{d_1}}] \in \mathbb{R}^{m \times (d_1+1)m}, \quad \bar{\mathbf{P}}_F := [\bar{P}_{F_0} \ \bar{P}_{F_1} \ \dots \ \bar{P}_{F_{d_1-1}} \ \bar{P}_{F_{d_1}}] \in \mathbb{R}^{m \times (d_1+1)m}.$$

Then for a fixed $\bar{P}_F(s) \in \mathcal{N}_s$, define the set:

$$\mathcal{M}_s := \{P_F(s) \in \mathbb{R}^{m \times m}[s] : \bar{\mathbf{P}}_F^T \mathbf{P}_F + \mathbf{P}_F^T \bar{\mathbf{P}}_F - \Pi^T (S \otimes T) \Pi \succ 0, \text{ for some } T = T^T \in \mathbb{R}^{d_1 m \times d_1 m}\} \quad (23)$$

where \otimes refers to the Kronecker product, $\succ 0$ implies a positive definite matrix, $S = [s_{11} \ s_{12}; s_{12} \ s_{22}] \in \mathbb{R}^{2 \times 2}$ (s_{ij} are as in (4)) and $\Pi \in \mathbb{R}^{2d_1 m \times (d_1+1)m}$ denotes a projection matrix given by

$$\Pi = \begin{bmatrix} I_m & 0 & \dots & 0 \\ & \ddots & & I_m \\ & & I_m & \ddots \\ 0 & \dots & 0 & I_m \end{bmatrix}^T$$

It was shown in (Henrion et al. 2003, Lemma 1) that for any given stable polynomial matrix $\bar{P}_F(s) \in \mathcal{N}_s$, the polynomial matrix $P_F(s) \in \mathcal{N}_s$ if there exists a symmetric matrix $T \in \mathbb{R}^{d_1 m \times d_1 m}$ satisfying the matrix inequality $\bar{\mathbf{P}}_F^T \mathbf{P}_F + \mathbf{P}_F^T \bar{\mathbf{P}}_F - \Pi^T (S \otimes T) \Pi \succ 0$. Hence, for every fixed $\bar{P}_F(s)$, this result characterizes a subset of the stable polynomial matrices and hence the set $\mathcal{M}_s \subseteq \mathcal{N}_s$. Then, by replacing the set \mathcal{N}_s with the approximated set \mathcal{M}_s we can formulate following problem.

Problem 4.2 Find $\inf \|F\|_F$ such that the polynomial matrix $P_F(s) \in \mathcal{M}_s$.

However, the matrix inequality $\bar{\mathbf{P}}_F^T \mathbf{P}_F + \mathbf{P}_F^T \bar{\mathbf{P}}_F - \Pi^T (S \otimes T) \Pi \succ 0$ in (23) is not yet an LMI in the unknown variables F and $U(s)$, which appear non-linearly in the polynomial matrix $P_F(s) = [\tilde{P}(s) + F\tilde{R}(s)] U(s)$. In the following section we will present a methodology to linearize the coefficients of $P_F(s)$.

4.2 Linearization of the coefficients of $P_F(s)$

Let us define a polynomial matrix $Y(s) := FR(s) \in \mathbb{R}^{m \times m}[s]$. Then, from the proof of Corollary 4.1, the degree of the polynomial matrix $Y(s)$ would be $d_1 - 1$ and hence we can write: $Y(s) = Y_0 + Y_1 s + \dots +$

$Y_{d_1-1}s^{d_1-1}$. Since $Y(s) = FR(s)$ we can write $P_F(s) = P(s) + Y(s)$. Hence

$$\begin{aligned} \mathbf{P}_F &= [P_{F_0} \ P_{F_1} \ \cdots \ P_{F_{d_1-1}} \ P_{F_{d_1}}] \\ &= [P_0 \ P_1 \ \cdots \ P_{d_1-1} \ P_{d_1}] + [Y_0 \ Y_1 \ \cdots \ Y_{d_1-1} \ \mathbf{0}] \end{aligned} \quad (24)$$

We have already shown in (20) that $[P_0 \ P_1 \ \cdots \ P_{d_1}] = \tilde{\mathbf{P}}\mathbf{U}$. Denoting $\mathbf{Y} = [Y_0 \ Y_1 \ \cdots \ Y_{d_1-1} \ \mathbf{0}]$, (24) can be written as

$$\mathbf{P}_F = \tilde{\mathbf{P}}\mathbf{U} + \mathbf{Y}. \quad (25)$$

By using (25), the matrix inequality in (23) would be

$$\tilde{\mathbf{P}}_F^T \tilde{\mathbf{P}}\mathbf{U} + \mathbf{U}^T \tilde{\mathbf{P}}^T \tilde{\mathbf{P}}_F + \tilde{\mathbf{P}}_F^T \mathbf{Y} + \mathbf{Y}^T \tilde{\mathbf{P}}_F - \Pi^T (S \otimes T) \Pi \succ 0 \quad (26)$$

which is an LMI with respect to the variables \mathbf{U} , \mathbf{Y} and T . However, the optimization variable F does not appear in this expression explicitly. In the following section we will rewrite F in terms of Y and U .

4.3 Relationship between F and $Y(s)$

Let us denote $y_j(s)$ as the j^{th} column of $Y(s) := FR(s)$. Following the discussion given in the proof of Corollary 4.1, it is clear that $\delta(y_j) = d_{j-1}$. Assume that the elements $y_{ij}(s)$ of $y_j(s)$ are in the following form:

$$y_{ij}(s) = y_{ij}^0 + y_{ij}^1 s + y_{ij}^2 s^2 + \cdots + y_{ij}^{d_j-1} s^{d_j-1} \quad (27)$$

for $i = 1, 2, \dots, m$. Furthermore, we have showed in Corollary 4.1 that the degree of the polynomial matrix $K(s) := W(s)U(s)$ would be $d_1 - 1$. Then, $K(s)$ can be written as

$$K(s) = K_0 + K_1 s + K_2 s^2 + \cdots + K_{d_1-1} s^{d_1-1} \quad (28)$$

We have already seen in Section 2.3 that $R(s) = Q^{-1}W(s)U(s)$. Hence the polynomial matrix $Y(s) = FR(s) = FQ^{-1}K(s)$ can be written as

$$Y(s) = FQ^{-1}K_0 + FQ^{-1}K_1 s + \cdots + FQ^{-1}K_{d_1-1} s^{d_1-1}. \quad (29)$$

Let us denote k_{z_j} as the j^{th} column of K_z in (28) for $z = 0, 1, \dots, d_1 - 1$. Let us denote f_i^T as the i^{th} row of the feedback gain matrix F . Then, from (27) and (29) it follows that the elements $y_{ij}(s)$ for a fixed column $y_j(s)$ for $j = 1, 2, \dots, m$ of $Y(s)$ would be:

$$\begin{aligned} y_{ij}^0 &= f_i^T Q^{-1} k_{0_j} = k_{0_j}^T (Q^{-1})^T f_i \\ y_{ij}^1 &= f_i^T Q^{-1} k_{1_j} = k_{1_j}^T (Q^{-1})^T f_i \\ &\vdots \\ y_{ij}^{d_j-1} &= f_i^T Q^{-1} k_{(d_j-1)_j} = k_{(d_j-1)_j}^T (Q^{-1})^T f_i \end{aligned} \quad (30)$$

for $i = 1, 2, \dots, m$. Let us construct following two matrices by using (30):

$$N^T = \begin{bmatrix} k_{0_1}^T \\ k_{1_1}^T \\ \vdots \\ k_{(d_1-1)_1}^T \\ k_{0_2}^T \\ k_{1_2}^T \\ \vdots \\ k_{(d_2-1)_2}^T \\ \vdots \\ k_{0_m}^T \\ k_{1_m}^T \\ \vdots \\ k_{(d_m-1)_m}^T \end{bmatrix} \quad \text{and} \quad \hat{y}_i = \begin{bmatrix} y_{i1}^0 \\ y_{i1}^1 \\ \vdots \\ y_{i1}^{d_1-1} \\ y_{i2}^0 \\ y_{i2}^1 \\ \vdots \\ y_{i2}^{d_2-1} \\ \vdots \\ y_{im}^0 \\ y_{im}^1 \\ \vdots \\ y_{im}^{d_m-1} \end{bmatrix} \quad (31)$$

Then, combining (30) and (31) we can write:

$$N^T(Q^{-1})^T f_i = \hat{y}_i \text{ for } i = 1, 2, \dots, m. \quad (32)$$

The relation (32) can be compactly written as:

$$f = [N_d G_d]^{-1} y \quad (33)$$

where

$$f = [f_1^T \ f_2^T \ \dots \ f_m^T]^T; \quad y = [\hat{y}_1^T \ \hat{y}_2^T \ \dots \ \hat{y}_m^T]^T; \\ N_d = \text{diag} \{N^T; N^T; \dots; N^T\}; \quad G_d = \text{diag} \{(Q^{-1})^T; (Q^{-1})^T; \dots; (Q^{-1})^T\}.$$

Then, according to (33), the cost function $\|F\|_F^2$ of Problem 4.2 would be

$$\|F\|_F^2 = f^T f = y^T [N_d G_d G_d^T N_d^T]^{-1} y \quad (34)$$

which is a non-convex function in the optimization variables y and N_d . In the following section we will overcome this difficulty by computing an upper bound for the cost function $\|F\|_F^2$ and then formulating an optimization in terms of the variables appearing in the resulting function.

4.4 Optimization Formulation

It is clear that $\|F\|_F^2 = \sum_{i=1}^m \sum_{j=1}^n |f_{ij}|^2 = \|f\|_2^2$. Hence minimization of $\|F\|_F$ is equivalent to minimizing $\|f\|_2$. However, according to (34), $\|f\|_2$ is non-convex in N_d and y . From (33) we have

$$\|f\|_2 = \|(N_d G_d)^{-1} y\|_2 \leq \|(N_d G_d)^{-1}\|_2 \|y\|_2. \quad (35)$$

To obtain convexity, we relax the requirements by minimizing the upper bound of $\|f\|_2$, i.e. $\|y\|_2$ and $\|(N_d G_d)^{-1}\|_2$ separately. This relaxed problem can be formulated as LMIs. To see this, note that (Horn

and Johnson 1991)

$$\|(N_d G_d)^{-1}\|_2 = \frac{1}{s_n(N_d G_d)}$$

where $s_n(N_d G_d)$ is the smallest singular value of $N_d G_d$. Hence $\|(N_d G_d)^{-1}\|_2$ can be minimized by maximizing $s_n(N_d G_d)$. An LMI formulation to maximize $s_n(N_d G_d)$ is discussed next.

Lemma 4.3: *If some N_d and positive scalar β satisfy*

$$\begin{bmatrix} \frac{1}{2} G_d^T (N_d + N_d^T) G_d & \beta I \\ \beta I & I \end{bmatrix} \succ 0 \tag{36}$$

then $s_n(N_d G_d)$ of $N_d G_d$ will satisfy

$$s_n(N_d G_d) \geq \frac{\beta^2}{s_1(G_d^T)} \tag{37}$$

where $s_1(G_d^T)$ is the largest singular value of G_d^T .

Proof See Appendix. □

It is clear from the relation (37) that we can maximize $s_n(N_d G_d)$ by maximizing the positive scalar β (since $s_1(G_d^T)$ is fixed according to the choice of Q).

Lemma 4.4: *If some N_d , y and positive scalars β , γ satisfy*

$$i) \begin{bmatrix} \frac{1}{2} G_d^T (N_d + N_d^T) G_d & \beta I \\ \beta I & I \end{bmatrix} \succ 0, ii) \begin{bmatrix} I & y \\ y^T & \gamma \end{bmatrix} \succ 0 \tag{38}$$

then the vector f will satisfy

$$\|f\|_2 \leq s_1(G_d^T) \frac{\sqrt{\gamma}}{\beta^2} \tag{39}$$

Proof According to the Lemma 4.3 we have

$$\frac{1}{s_n(N_d G_d)} \leq \frac{s_1(G_d^T)}{\beta^2}.$$

Hence

$$\|(N_d G_d)^{-1}\|_2 = \frac{1}{s_n(N_d G_d)} \leq \frac{s_1(G_d^T)}{\beta^2}.$$

Furthermore, using the Schur complement in LMI (ii) we have $\|y\|_2^2 \leq \gamma$. Recall that

$$\begin{aligned} \|f\|_2 &= \|(N_d G_d)^{-1} y\|_2 \leq \|(N_d G_d)^{-1}\|_2 \|y\|_2 \\ &\leq s_1(G_d^T) \frac{\sqrt{\gamma}}{\beta^2} \end{aligned}$$

This completes the proof. □

Recall that the polynomial matrix $U(s) \in \mathbb{R}^{m \times m}[s]$, constructed according to Theorem 3.2, will be unimodular if and only if the diagonal elements of the lower triangular matrix $U_{11}(s)$ are non zero and

the real matrix U_{22} is nonsingular. To achieve this, let us construct a block diagonal real matrix U_D from the polynomial matrix (18), with the sub-blocks corresponding to U_{kt} of the form:

$$U_{D_{kt}} := \left[\begin{array}{c|c} U_{D_{11}} & \mathbf{O} \\ \hline \mathbf{O} & U_{D_{22}} \end{array} \right] \quad (40)$$

where $U_{D_{11}}$ is a diagonal matrix with diagonal elements equal to the diagonal elements of $U_{11}(s)$ and $U_{D_{22}} = U_{22}$. Then following result holds.

Theorem 4.5: *Let $\tilde{\mathbf{P}}, \mathbf{U}, \mathbf{Y}$ be the real matrices corresponding to the polynomial matrices $\tilde{P}(s), U(s)$ and $Y(s)$ respectively. Let us consider a fixed polynomial matrix $\tilde{P}_F(s) \in \mathcal{N}_s$. If for some $N_d, \mathbf{U}, U_D, \mathbf{Y}, y$ and positive scalars β, γ , following conditions:*

$$\begin{aligned} & i) \frac{1}{2} (U_D + U_D^T) \succ 0 \\ & ii) \tilde{\mathbf{P}}_F^T \tilde{\mathbf{P}} \mathbf{U} + \mathbf{U}^T \tilde{\mathbf{P}}^T \tilde{\mathbf{P}}_F + \tilde{\mathbf{P}}_F^T \mathbf{Y} + \mathbf{Y}^T \tilde{\mathbf{P}}_F - \Pi^T (S \otimes T) \Pi \succ 0 \\ & iii) \left[\begin{array}{cc} \frac{1}{2} G_d^T (N_d + N_d^T) G_d & \beta I \\ \beta I & I \end{array} \right] \succ 0, \quad \left[\begin{array}{c} I \ y \\ y^T \ \gamma \end{array} \right] \succ 0 \end{aligned}$$

are satisfied then $U(s)$ is a unimodular polynomial matrix and all the zeros of the polynomial matrix $P_F(s)$ are confined within the stability region \mathbb{S} . Furthermore, the Frobenius norm of F will satisfy

$$\|F\|_F \leq s_1(G_d^T) \frac{\sqrt{\gamma}}{\beta^2} \quad (41)$$

Proof Since $U(s)$ is a block triangular matrix, it directly follows from the condition (i) that the polynomial matrix $U(s)$ is a unimodular matrix. Condition (ii) directly follows from (26) implying that the polynomial matrix $P_F(s) \in \mathcal{M}_s$. Since $\mathcal{M}_s \subseteq \mathcal{N}_s$, the polynomial matrix $P_F(s) \in \mathcal{N}_s$. This implies all the zeros of $P_F(s)$ are in \mathbb{S} . Furthermore, since N_d, y and positive scalars β, γ satisfy conditions in (iii), the Frobenius norm of F will satisfy the inequality in (41) which directly follows from Lemma 4.4. \square

Recall that we are interested in finding a minimum norm feedback gain matrix with which the regional pole placement can be done. Since $\|F\|_F \leq s_1(G_d^T) \frac{\sqrt{\gamma}}{\beta^2}$, we can indirectly minimize the $\|F\|_F$ by formulating the following optimization problem:

Problem 4.6

$$\max_{u_{ij}, y_{ij}^k, \beta, \gamma, T} \beta - \gamma$$

such that

$$\begin{aligned} & i) \frac{1}{2} (U_D + U_D^T) \succ 0 \\ & ii) \tilde{\mathbf{P}}_F^T \tilde{\mathbf{P}} \mathbf{U} + \mathbf{U}^T \tilde{\mathbf{P}}^T \tilde{\mathbf{P}}_F + \tilde{\mathbf{P}}_F^T \mathbf{Y} + \mathbf{Y}^T \tilde{\mathbf{P}}_F - \Pi^T (S \otimes T) \Pi \succ 0 \\ & iii) \left[\begin{array}{cc} \frac{1}{2} G_d^T (N_d + N_d^T) G_d & \beta I \\ \beta I & I \end{array} \right] \succ 0, \quad \left[\begin{array}{c} I \ y \\ y^T \ \gamma \end{array} \right] \succ 0 \end{aligned}$$

where β and γ are positive scalars.

Note that in Problem 4.6 the variables in matrices \mathbf{U}, N_d and U_d are u_{ij} 's. Similarly in matrix \mathbf{Y} and vector y the variables are y_{ij}^k 's. Moreover, for a choice of $\tilde{P}_F(s) \in \mathcal{N}_s$, all the constraints in Problem 4.6 are linear with respect to the optimization variables $(\mathbf{U}, \mathbf{Y}, U_D, T, N_d, y, \beta, \gamma)$. Hence it is an LMI optimization problem and can be solved with solvers like *SeDuMi*. After solving Problem 4.6, the feedback gain matrix F can be computed from the relation (32).

4.5 Fixed Pole Placement

In this section we briefly outline the application of the proposed method for conventional fixed pole placement, where we find $\inf \|F\|_F$ such that the eigenvalues of $(A - BF)$ are placed at specific locations (pre-decided) in the complex plane.

We have already seen that the closed loop poles are equal to the roots of the polynomial $|P_F(s)|$. Hence to place the closed loop poles at specific locations in the complex plane we have to first construct a polynomial matrix $P_F(s)$ such that the zeros of $P_F(s)$ are equal to the closed loop poles. However, for a given set of closed loop poles there are infinitely many $P_F(s)$ and this non uniqueness, is characterized by the unimodular polynomial matrices in Section 3. We consider unimodular polynomial matrices $V(s)$ such that $(P_F(s)V(s))$ is column reduced and its column degrees are equal to the column degrees of $P_F(s)$. Since $P_F(s) = P(s) + FR(s)$ and the zeros of $P_F(s)$ are equal to the zeros of $(P_F(s)V(s))$, we will use $P_F(s)V(s) = P(s) + FR(s)$ as a constraint for the optimization problem. Let us define a matrix \mathbf{V} similar to the definition of \mathbf{U} in (21). Furthermore, let $\mathbf{P}_F := [P_{F_0} \ P_{F_1} \ \cdots \ P_{F_{d_1}}]$ where P_{F_k} for $k = 0, 1, \dots, d_1$ are the coefficients of the designed $P_F(s)$ (zeros are equal to the closed loop poles). Then, we can write $\mathbf{P}_F \mathbf{V} = \tilde{\mathbf{P}} \mathbf{U} + \mathbf{Y}$. Hence following the developments in Section 4.2, 4.3 and 4.4 we can minimize an upper bound of $\|F\|_F$ while placing the poles at fixed locations, given in $P_F(s)$, through the following optimization:

Problem 4.7

$$\max_{u_{ij}, v_{ij}, y_{ij}^k, \beta, \gamma} \beta - \gamma$$

such that

$$\begin{aligned} & i) \frac{1}{2} (U_D + U_D^T) \succ 0 \\ & ii) \tilde{\mathbf{P}} \mathbf{U} + \mathbf{Y} = \mathbf{P}_F \mathbf{V} \\ & iii) \begin{bmatrix} \frac{1}{2} G_d^T (N_d + N_d^T) G_d & \beta I \\ \beta I & I \end{bmatrix} \succ 0, \quad \begin{bmatrix} I & y \\ y^T & \gamma \end{bmatrix} \succ 0 \end{aligned}$$

where β and γ are positive scalars.

Note that all the constraints in above problem are linear in optimization variables $(\mathbf{U}, \mathbf{V}, \mathbf{Y}, U_D, N_d, y, \beta, \gamma)$. Hence it is an LMI optimization problem.

In the following section we demonstrate the proposed algorithms through numerical examples.

5 Numerical Examples

Example 1: Let us consider a distillation column plant where

$$A = \begin{bmatrix} -0.1094 & 0.0628 & 0 & 0 & 0 \\ 1.306 & -2.132 & 0.9807 & 0 & 0 \\ 0 & 1.595 & -3.149 & 1.547 & 0 \\ 0 & 0.0355 & 2.632 & -4.257 & 1.855 \\ 0 & 0.00227 & 0 & 0.1636 & -0.1625 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0 \\ 0.0638 & 0 \\ 0.0838 & -0.1396 \\ 0.1004 & -0.2060 \\ 0.0063 & -0.0128 \end{bmatrix}.$$

The controllability indices are $d_1 = 3$ and $d_2 = 2$. The eigenvalues of A are at $-5.9822, -2.8408, -0.8953, -0.0142,$ and -0.0773 . This example was used in (Kautsky et al. 1985) to illustrate the widely used pole placement algorithm implemented as the *place* command in MATLAB. We use the method developed above to synthesize a low norm state feedback controller for this system while improving the plant settling time. As mentioned in the introduction no available algorithm allows for feedback norm optimization with regional pole placement. Hence the results are compared with the norm minimization

results reported in (Varga 2000a), and the *place* algorithm results from (Kautsky et al. 1985), both of which can address only fixed pole placement requirements.

Regional Pole Placement: Assume that all the closed loop poles need to be placed to the left of the vertical line at -0.2 in the complex plane. Corresponding to this stability region the matrix S in (4) would be $S = \begin{bmatrix} 0.4 & 1 \\ 1 & 0 \end{bmatrix}$. Since $d_1 = 3$ and $d_2 = 2$ the unimodular polynomial matrix, following Section 3, would be

$$U(s) = \begin{bmatrix} u_{11}^0 & 0 \\ u_{21}^1 s + u_{21}^0 & u_{22}^0 \end{bmatrix}$$

and hence $K(s) = W(s)U(s)$ would be

$$K(s) = \begin{bmatrix} u_{11}^0 & 0 \\ su_{11}^0 & 0 \\ s^2 u_{11}^0 & 0 \\ u_{21}^1 s + u_{21}^0 & u_{22}^0 \\ u_{21}^1 s^2 + u_{21}^0 s & u_{22}^0 s \end{bmatrix}.$$

Then, by defining a polynomial matrix

$$Y(s) := \begin{bmatrix} y_{11}^0 + y_{11}^1 s + y_{11}^2 s^2 & y_{12}^0 + y_{12}^1 s \\ y_{21}^0 + y_{21}^1 s + y_{21}^2 s^2 & y_{22}^0 + y_{22}^1 s \end{bmatrix}$$

and following the procedure discussed in Section 4.3, (31) will take the following form:

$$N^T = \begin{bmatrix} u_{11}^0 & 0 & 0 & u_{21}^0 & 0 \\ 0 & u_{11}^0 & 0 & u_{21}^1 & u_{21}^0 \\ 0 & 0 & u_{11}^0 & 0 & u_{21}^1 \\ 0 & 0 & 0 & u_{22}^0 & 0 \\ 0 & 0 & 0 & 0 & u_{22}^0 \end{bmatrix}; \hat{y}_1 = [y_{11}^0 \ y_{11}^1 \ y_{11}^2 | y_{12}^0 \ y_{12}^1]^T; \hat{y}_2 = [y_{21}^0 \ y_{21}^1 \ y_{21}^2 | y_{22}^0 \ y_{22}^1]^T$$

Solving Problem 4.6, the following optimal feedback gain matrix

$$F = \begin{bmatrix} 6.9055 & 1.1598 & -3.4003 & 1.8236 & 6.9800 \\ 29.7580 & -3.3597 & 7.0876 & -4.3327 & -4.1177 \end{bmatrix}$$

is obtained. Corresponding to this optimal F , the closed loop poles are at -6.2978 , -2.5402 and -0.5436 , $-0.2001 \pm 0.0035i$. It is clear that all the closed loop poles are placed within the designed stability region and hence the objective of improving the settling time is achieved. The optimal norms of the feedback gain matrix is shown in Table 1.

We compare our results with an existing norm minimization algorithm proposed in (Varga 2000a), which however do not allow for regional pole placement. Hence in that method, the desired closed loop poles needs to be chosen a priori. The optimal F , designed according to (Varga 2000a) for fixed closed loop poles: $-0.2, -0.5, -1, -1 \pm 1i$, is given in Table 1. It is observed that around 65.7536% reduction in $\|F\|_2$ (in proposed algorithm $\|F\|_2 = 31.7019$) is achieved in comparison to the result reported in (Varga 2000a) mainly due to the flexibility afforded by the regional placement capability of the proposed algorithm. In addition to this, we compare our results with the conventional fixed pole placement algorithm by (Kautsky et al. 1985). The feedback matrix F is computed by *place* command in MATLAB corresponding to the closed loop poles: $-0.2, -0.5, -1, -1 \pm 1i$. We observe 92% reduction in $\|F\|_F$ with the proposed algorithm in comparison to *place* algorithm in MATLAB.

However, our algorithm does not perform necessarily better than (Varga 2000a) for the same fixed closed loop locations. The various approximations for obtaining a convex problem leads to a suboptimal solution for fixed poles. This can be seen if we compute the optimal F corresponding to the fixed closed

Table 1. Comparison Table

Constraints	Algorithm Name	Norms
Regional Pole Placement	Proposed Algo.	$\ F\ _F = 33.0972$ $\ F\ _2 = 31.7019$
Closed loop poles: $-0.2, -0.5, -1, -1 \pm 1i$	Varga's Algo.	$\ F\ _2 = 92.5700$
	<i>place</i> in MATLAB	$\ F\ _F = 413.8233$

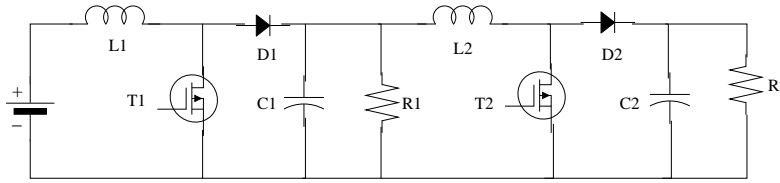


Figure 1. A Boost-Boost converter, obtained by cascade connection of two boost converters. It has two control switches T_1 and T_2 and hence it is a multi-variable DC to DC converter. R_1 and R_2 are two loads.

loop poles: $-0.2, -0.5, -1, -1 \pm 1i$ by solving Problem 4.7. The resulting feedback gain matrix is

$$F = \begin{bmatrix} 30.0088 & -40.8940 & 54.3899 & -32.236 & 12.749 \\ 18.1854 & 16.1808 & -60.3002 & 66.076 & -22.757 \end{bmatrix}$$

and its norm is $\|F\|_2 = 117.7246$, as compared to $\|F\|_2 = 92.5700$ obtained for these pole locations in (Varga 2000a). Our algorithm still easily outperforms the *place* algorithm ($\|F\|_F = 413.8233$), but the main advantage of our algorithm is that the closed loop poles do not need to be (arbitrarily) specified. The poles are chosen automatically by the algorithm to produce the lowest feedback norm, which in this case is ($\|F\|_2 = 31.7019$) considerably lower than the other algorithms.

Example 2: In this example we consider a problem of designing linear state feedback controller for a DC to DC (direct current) Boost converter. Since Boost power converters are capable of magnifying the level of input voltage across the output, they play a vital role in several applications like hybrid electric vehicles (Xu et al. 2005) and distributed energy sources (Dobbs and Chapman 2003). A cascade connection of two Boost converters, known as *Boost-Boost* converter, is shown in Fig. 1. An average normalized linear model (see (Sira-Ramirez and Silva-Ortigoza 2006, Chapter 2 and 4)) for this converter is given by following matrices:

$$A = \begin{bmatrix} 0 & -0.6667 & 0 & 0 \\ 0.6667 & -2.0000 & -1.0000 & 0 \\ 0 & 1.0000 & 0 & -0.7500 \\ 0 & 0 & 0.7500 & -1.3333 \end{bmatrix}; \quad B = \begin{bmatrix} -1.5000 & 0 \\ 9.8333 & 0 \\ 0 & -2.0000 \\ 0 & 3.5556 \end{bmatrix}.$$

It is observed that the system is completely controllable with controllability indices $d_1 = d_2 = 2$ and hence we can place the closed loop poles at arbitrary locations in the complex plane. A low magnitude linear state feedback controller is designed such that all the closed loop poles are placed left to the vertical line at -0.8 in the complex plane to achieve 5 second settling time. Corresponding to this stability region we have $s_{11} = 1.6, s_{12} = 1$ and $s_{22} = 0$. Solving Problem 4.6, the resulting feedback gain matrix:

$$F = \begin{bmatrix} -0.0804 & -0.0207 & -0.0532 & -0.0035 \\ -0.0395 & -0.0105 & -0.0272 & 0.0163 \end{bmatrix}$$

and the corresponding closed loop poles are $-0.8565 \pm 0.9530i, -0.8445, -0.8051$. The Frobenius norm of the resulting feedback matrix F is given in Table 2. To compare the results with conventional approach, we compute feedback gain matrix by *place* command in MATLAB. The resulting gain matrix corresponding to the above optimal closed loop poles is

$$F_{conv} = \begin{bmatrix} 0.0953 & -0.2119 & -0.1928 & 0.0731 \\ 0.5015 & -0.9324 & -0.4795 & 0.3647 \end{bmatrix}.$$

Comparing the norm of F with F_{conv} , it is clear that there is around 91.13% reduction in norm of the

Table 2. Comparison Table

Algorithm Name	$\ F\ _F$	% reduction
Proposed Algorithm	0.1114	91.1383
place in MATLAB	1.2571	-

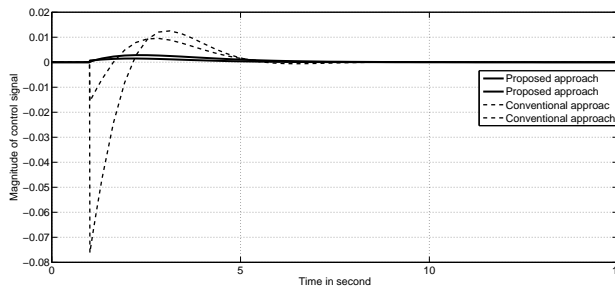


Figure 2. Comparison between the maximum overshoot of the controller effort. The magnitude of maximum overshoots of $|u_{1av}(t)|$ with conventional (dashed line) and proposed (solid line) approach are 0.01555 and 0.00283 respectively. Furthermore, $\max_t |u_{2av}(t)|$ with conventional (dashed lines) and proposed (solid lines) approach are 0.07515 and 0.00147 respectively. The percentage reduction in $\max_t |u_{1av}(t)|$ and $\max_t |u_{2av}(t)|$ are 81.87% and 98.04% respectively in proposed approach.

feedback gain matrix with the proposed approach.

To compare the controller effort ($|u(t)|$) between the conventional pole placement approach and the proposed approach, we consider following two cases: the converter is driven by implementing i) F in the control circuit and ii) F_{conv} in the control circuit. Simulation for both the cases are done in MATLAB Simulink and the control signals u_{1av} and u_{2av} are shown in Fig. 2. We observe more than 80% reduction in maximum overshoot of the controller effort ($\max_t |u(t)|$) in both control signals. Furthermore, the response of states corresponding to the feedback gains F and F_{conv} is depicted in Fig. 5 and is observed that all states are settled within 5 second.

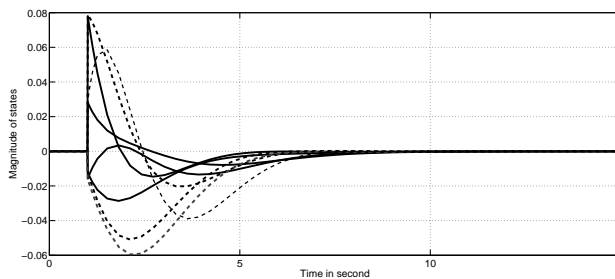


Figure 3. Response of the states of Boost-Boost converter which is driven by implementing the feedback gain matrix F obtained by proposed approach. The simulation is done in MATLAB Simulink. The closed loop system (consisting of linearized model of Boost-Boost converter and state feedback gain matrix F (solid lines) and F_{conv} (dashed lines)) is excited with a step input of step length 0.01 second.

6 Conclusion

This article develops a convex method to minimize an upper bound of the norm of state feedback gain matrix while achieving regional pole placement specifications on closed loop poles. ~~The proposed algorithm is compared with the existing norm minimization algorithm proposed by Varga.~~ Apriori choices for closed loop poles are difficult to make for the designer often leading to high feedback gains, even with modern norm minimizing algorithms. ~~In addition, the existing norm minimization algorithms (for instance the algorithm proposed by Varga) do not allow regional pole placement objective.~~ ~~Chilali et al. have proposed algorithms of designing F to minimize the H_∞ and/or H_2 norm of the closed loop system while guaranteeing regional pole placement.~~

In this work we combine the objective of finding a low gain feedback matrix with achieving regional pole placement. The main advantage of the proposed method lies in not having to pre-specify the closed

loop poles, thereby achieving low feedback gains, resulting in lower actuation efforts. The proposed algorithm can be improved on several fronts: to obtain a convex formulation we minimize an upper-bound of the norm rather than the norm itself, thereby making the method suboptimal. Furthermore, the central polynomial matrix $\bar{P}_F(s)$, which determines the stability region in the coefficient space, has to be chosen heuristically. While these topics are under current investigation, the current algorithm seems to perform well in most numerical situations.

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Appendix

Let us denote $A_s = \frac{1}{2}(A + A^T)$ as the symmetric part of a matrix $A \in \mathbb{R}^{n \times n}$.

Corollary 6.1: (Horn and Johnson 1991, Corollary 3.1.5) Let $A \in \mathbb{R}^{n \times n}$ and denote $s_1(A) \geq s_2(A) \geq \dots \geq s_n(A)$ as the singular values of A . Assume that the eigenvalues, denoted as $\lambda_k(A_s)$, of symmetric matrix A_s are arranged as follows: $\lambda_1(A_s) \geq \lambda_2(A_s) \geq \dots \geq \lambda_n(A_s)$. Then we have

$$s_k(A) \geq \lambda_k(A_s) \text{ for } k = 1, 2, \dots, n. \quad (42)$$

The following inequality gives a bound on the singular values of the product of two matrices (see (Horn and Johnson 1991, Theorem 3.3.16)).

Theorem 6.2: Let us consider two matrices $A, B \in \mathbb{R}^{n \times n}$. Then following inequality holds

$$s_{i+j-1}(AB) \leq s_i(A)s_j(B) \quad (43)$$

for $1 \leq i, j \leq n$ and $i + j \leq n + 1$. In particular

$$s_n(AB) \leq s_1(A)s_n(B) \quad (44)$$

Proof of Lemma 4.3:

Proof According to the Schur complement formula the LMI in (36) is equivalent to

$$\begin{aligned} & \frac{1}{2}G_d^T(N_d + N_d^T)G_d - \beta^2 I \succ 0 \\ \Rightarrow & \lambda_{\min}\left(\frac{1}{2}(G_d^T(N_d + N_d^T)G_d)\right) \geq \beta^2 \end{aligned} \quad (45)$$

where $\lambda_{\min}(G_d^T(N_d + N_d^T)G_d)$ is the smallest eigenvalue of $G_d^T(N_d + N_d^T)G_d$. But according to the relation (42) we have

$$s_n(G_d^T N_d G_d) \geq \lambda_{\min}\left(\frac{1}{2}(G_d^T(N_d + N_d^T)G_d)\right).$$

Furthermore, according to the relation (44), we have

$$s_1(G_d^T) s_n(N_d G_d) \geq s_n(G_d^T N_d G_d).$$

Using this in relation (45) we have

$$s_1(G_d^T)s_n(N_dG_d) \geq \beta^2$$

and hence we have the desired relation given in (37). \square

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