

# COMPUTATION OF STATE REACHABLE POINTS OF SECOND ORDER LINEAR TIME-INVARIANT DESCRIPTOR SYSTEMS\*

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**Abstract.** This paper considers the problem of computing the state reachable points (from the origin) of a linear constant coefficient second order descriptor system. A new method is proposed to compute the reachable set in a numerically stable way. The original descriptor system is transformed into a strangeness-free system within the behavioral framework followed by a projection that separates the system into different order differential and algebraic equations while keeping the original state variables. This reformulation is followed by a first order formulation that avoids all unnecessary smoothness requirements. For the resulting first order system it is shown that the computation of the image space of two matrices, associated with the projected system, is enough to numerically compute the reachable set. Moreover, a characterization is presented of all the inputs by which one can reach an arbitrary point in the reachable set. These results are used to compute two different types of reachable sets for second order systems. The new approach is demonstrated through a numerical example.

**Key words.** Linear time-invariant descriptor system, behavior formulation, strangeness-free formulation, reachability, derivative array, second order system.

**AMS subject classifications.** 93C05, 93C15, 93B05

**1. Introduction.** Due to the wide availability of automated modeling tools [1, 18, 25] the modeling of dynamical systems via descriptor systems, where the system equation is a differential-algebraic equation (DAE), has become industrial standard in many application domains, see e.g. [4, 12, 21, 30, 33, 34].

Modeling with descriptor systems has the major advantage that constraints which restrict the dynamics of the system can be explicitly incorporated in the model. This is important, since they usually describe real physical properties like conservation laws, symmetries, or other invariants of the system. Further constraints may be implicitly hidden in the system that can only be revealed via further differentiation. This leads to a major distinction to ordinary state-space systems in the sense that descriptor systems not only integrate but also differentiate, see [21]. The number of required differentiations is usually described by an *index*, see [28] for a comparison of different index concepts. If in descriptor systems the inputs are not sufficiently smooth, then impulsive responses may arise [11, 12, 21, 32, 37] and a regularization process may be needed, see [10]. This is important, in particular, in the context of control systems, where typically the inputs are very often discontinuous, like in bang-bang control [2].

In this paper we consider numerical methods to detect *reachability*, i.e., whether a control input exists which transfers from one state to another state in finite time, for second order linear descriptor systems of the form

$$(1) \quad M\ddot{x} + D\dot{x} + Kx = Bu,$$

where  $M, D, K \in \mathbb{R}^{n,n}$ ,  $B \in \mathbb{R}^{n,m}$ ,  $x : \mathbb{R} \rightarrow \mathbb{R}^n$  is the state and  $u : \mathbb{R} \rightarrow \mathbb{R}^m$  is an input (control) to the system. Furthermore, initial conditions  $x(t_0) = x_0$  and  $\dot{x}(t_0) = v_0$

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are assumed to be available. We assume that the system is *regular*, i.e., there exists a complex number  $s$  such that  $s^2M + sD + K$  is invertible, but more general situations are possible, see [29, 35]. We introduce numerical methods to compute the set of all *state reachable points* via some chosen set of input functions. In particular, we show that the reachability sets are characterized by the image space of certain matrices.

The classical textbook approach to study these problems would be to turn the second order system into first order by introducing a new variable  $v = \dot{x}$  and then treat the reachability questions for the first order case. But this is somewhat subtle for descriptor systems.

EXAMPLE 1. Consider the second order descriptor system from [8],

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \ddot{x} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u,$$

which has the solution  $x_1 = b_2u$ ,  $x_2 = b_1u - b_2(u + \dot{u} + \ddot{u})$ . If  $b_2 \neq 0$  and  $\ddot{u}$  exists, that is,  $u$  is twice continuously differentiable, then the solution is continuous, but if we introduce a new variable  $v = \dot{x}$  and transfer the second order system to first order, then the solution is

$$\begin{aligned} x_1 &= b_2u, & x_2 &= b_1u - b_2(u + \dot{u} + \ddot{u}), \\ v_1 &= b_2\dot{u}, & v_2 &= b_1\dot{u} - b_2(\dot{u} + \ddot{u} + u^{(3)}), \end{aligned}$$

and hence for a continuous solution,  $u$  has to be three times continuously differentiable. One way to avoid this dilemma of extra smoothness requirements, and this is the approach that we follow below, is to only introduce  $v_1 = b_2\dot{u}$  as a new variable, see [8], where this process is called *trimmed first order formulation*.

Another difficulty in second order systems is that one has to distinguish between the set of states that can be reached for the second order formulation and that of a first order formulation, see also [24].

DEFINITION 2. Consider the second order descriptor system (1). A vector  $x_1 \in \mathbb{R}^n$  is said to be reachable from the initial condition  $x(t_0) = x_0$ , if there exists a (sufficiently smooth) control input  $u$ , an initial value for the derivative  $\dot{x}(t_0) = v(t_0)$ , and a finite time  $t_1 > 0$  such that  $x(t_1) = x_1$ .

A first order formulation of (1) with an extended state vector  $\xi_1 := [\tilde{v}_1^T, x_1^T]^T \in \mathbb{R}^{\tilde{n}+n}$  is said to be reachable from the initial condition  $\xi(t_0) := [\tilde{v}(t_0)^T, x(t_0)^T]^T \in \mathbb{R}^{\tilde{n}+n}$ , if there exists a (sufficiently smooth) control input  $u$  and a finite time  $t_1 > 0$  such that  $\xi(t_1) = \xi_1$ .

It is obvious that the two reachability sets are different.

EXAMPLE 3. A simple second order control system is the model of a two-dimensional, linearized model of a three-link mobile manipulator from [20], see also [5, 10], which has the form

$$M_0\ddot{\phi} + D_0\dot{\phi} + K_0\phi = F_0^T\lambda + S_0u, \quad F_0\phi = 0,$$

where  $M_0, D_0, K_0$  are mass, damping, and stiffness matrix,  $\lambda$  is a Lagrange multiplier, and  $F_0^T\lambda$  is the generalized constraint force. Setting  $x = [\phi^T \quad \lambda^T]^T$ , we obtain the second order descriptor system

$$\begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix} \ddot{x} + \begin{bmatrix} D_0 & 0 \\ 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} K_0 & -F_0^T \\ F_0 & 0 \end{bmatrix} x = \begin{bmatrix} S_0 \\ 0 \end{bmatrix} u.$$

Avoiding extra regularity one can use the trimmed first order formulation [8] and introduce  $\nu = \dot{\phi}$ . Since  $\phi$  and  $\nu$  are restricted by the cleaning surface, it makes a major difference whether one wants to reach a state  $\phi_1$  and does not care about the velocity  $\nu$  or whether one wants to reach  $\phi_1$  with a given velocity  $\nu_1$ .

The paper is organized as follows. In Section 2 we briefly recall the *derivative array* approach for first order systems to obtain a regularized model associated with the original descriptor system in which all constraints are explicitly available. Using the regularized system we define two projection matrices to separate the descriptor system additively into its differential and algebraic parts. Based on this decomposition, we derive a method to numerically compute the state reachable points of the descriptor system in Section 3. The results are then extended to second order systems in Section 4. A numerically stable regularization procedure is presented in Section 5 and based on this procedure the reachable sets are characterized. The results are illustrated by a numerical example from mechanical multi-body systems.

**2. Regularization of descriptor systems.** A traditional way to carry out the analysis and regularization of descriptor systems is to transform the system into an ordinary state space system by resolving all the algebraic constraint equations. However, in this way typically the variables may lose their physical interpretation and what is worse, the algebraic constraints are not available any longer in the system, and in numerical simulation or control, due to discretization and rounding errors, these constraints are violated, see the discussion in [10, 21, 22]. This approach gives particularly bad results when the solution of the algebraic equations is sensitive to perturbations.

Another classical approach, presented, e.g. in [12, 21, 27], is to use canonical forms to identify the invariants and to decouple the original system into the fast (algebraic) and slow (differential) subsystem. But the transformation to these canonical forms may be arbitrarily ill-conditioned [5, 21], so that small perturbations lead to physically unrealistic subsystems. But since the canonical forms are typically not numerically computable, this approach is of a more theoretical value. Partially, these difficulties can be avoided using staircase forms [15, 36] under orthogonal transformations, see [5] for a survey. However, also the computation of these staircase forms is very subtle in finite precision arithmetic, a complete error analysis is not available, and the physical interpretation of variables is changed.

To avoid the described difficulties, many approaches have been suggested for index reduction and reformulation of differential-algebraic systems, see [21, 23]. A very successful procedure in practice is the *derivative array approach*, see [9, 10, 21]. In this approach, since derivatives are needed, one adds a sufficient number of derivatives of the original model equations to the system, so that all necessary derivative information is available. From the resulting over-determined system then, via orthogonal projections, one identifies and separates the differential and algebraic equations. The derivative array approach is very robust to perturbations, and it has been implemented successfully in numerical simulation codes for linear and nonlinear differential-algebraic systems [21]. Another surplus of this approach is that the physical meaning of all the variables is preserved.

Let us briefly recall the derivative array approach for first order systems of the form

$$(2) \quad E\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$

where,  $E \in \mathbb{R}^{n,n}$ ,  $A \in \mathbb{R}^{n,n}$ , and  $B \in \mathbb{R}^{n,m}$ ,  $x : \mathbb{R} \rightarrow \mathbb{R}^n$  is the state and  $u : \mathbb{R} \rightarrow \mathbb{R}^m$  is

the input (control). Assuming that the system is regular, i.e., there exists a complex number  $s$  such that  $sE - A$  is invertible, see [10] for the general case, we express the state equation in a behavior framework, see [31], i.e., rewrite (2) as

$$(3) \quad \mathbf{E}\dot{z} = \mathbf{A}z,$$

where  $\mathbf{E} := [E \ 0] \in \mathbb{R}^{n, (n+m)}$  and  $\mathbf{A} := [A \ B] \in \mathbb{R}^{n, (n+m)}$ , by introducing  $z = [x^T \ u^T]^T$ . By performing a sequence of differentiations of (3) and stacking the original system and its derivatives up to order  $\mu$  on top of each other, we get a derivative array  $\mathcal{M}_\mu \dot{z}_\mu = \mathcal{N}_\mu z_\mu$ , where

$$\mathcal{M}_\mu = \begin{bmatrix} \mathbf{E} & 0 & \cdots & 0 \\ -\mathbf{A} & \mathbf{E} & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\mathbf{A} & \mathbf{E} \end{bmatrix}, \quad \mathcal{N}_\mu = \begin{bmatrix} \mathbf{A} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad z_\mu = \begin{bmatrix} z \\ \dot{z} \\ \vdots \\ z^{(\mu)} \end{bmatrix}.$$

To obtain a regularized system we use the following theorem, see [21].

**THEOREM 4.** *There exist integers  $\mu$ ,  $a$ , and  $d$  such that the inflated pair  $(\mathcal{M}_\mu, \mathcal{N}_\mu)$  associated with the given pair  $(\mathbf{E}, \mathbf{A})$  has the following properties:*

1.  $\text{rank} \mathcal{M}_\mu = (\mu + 1)n - a$  such that there exists a matrix  $Z_2$  with orthogonal columns of size  $((\mu + 1)n, a)$  satisfying  $Z_2^T \mathcal{M}_\mu = 0$ .
2.  $\text{rank} A_2 = a$ , where  $A_2 = Z_2^T \mathcal{N}_\mu [I_n \ 0 \ \cdots \ 0]^T$  such that there exists a matrix  $T_2$  with orthogonal columns of size  $(n, d)$ ,  $d = n - a$  satisfying  $A_2 T_2 = 0$ .
3.  $\text{rank} \mathbf{E} T_2 = d$  such that there exists a matrix  $Z_1$  with orthogonal columns of size  $(n, d)$  satisfying  $\text{rank} E_1 T_2 = d$  with  $E_1 = Z_1^T \mathbf{E}$ .

The smallest  $\mu \geq 0$  for which these conditions hold is called the *strangeness-index* of (3). Using the orthogonal matrices  $Z_1$ ,  $Z_2$  and  $T_2$ , we obtain a strangeness-free ( $\mu = 0$ ) system in behavior form associated with (3) given by

$$(4) \quad \begin{bmatrix} \widehat{E}_1 \\ 0 \end{bmatrix} \dot{z} = \begin{bmatrix} \widehat{A}_1 \\ \widehat{A}_2 \end{bmatrix} z$$

where the coefficients are given by

$$\widehat{E}_1 = Z_1^T \mathbf{E}, \quad \widehat{A}_1 = Z_1^T \mathbf{A}, \quad \widehat{A}_2 = Z_2^T \mathcal{N}_\mu \begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix}.$$

Systems (3) and (4) have the same solution set in the behavior sense and can be represented as  $\mathcal{E}\dot{x} = \mathcal{A}x + \mathcal{B}u$ , or

$$(5a) \quad E_1 \dot{x} = A_1 x + B_1 u,$$

$$(5b) \quad 0 = A_2 x + B_2 u,$$

respectively, where  $E_1 = Z_1^T \mathbf{E} \in \mathbb{R}^{d, n}$  has full row rank,

$$A_1 = \widehat{A}_1 \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad A_2 = \widehat{A}_2 \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad B_1 = \widehat{A}_1, \quad B_2 = \widehat{A}_2 \begin{bmatrix} 0 \\ I_m \end{bmatrix}.$$

Note that (5) may not be strangeness-free as a free system with  $u = 0$ , but it satisfies  $\text{rank}[\mathcal{E}, \mathcal{A}S_\infty, \mathcal{B}] = n$ , where the columns of the matrix  $S_\infty$  span the kernel of  $\mathcal{E}$ , i.e., it

is *impulse controllable* (or *controllable at infinity*), see [5, 12], and there exists a state feedback  $u = Fx$  such that the closed loop system  $\mathcal{E}\dot{x} = (\mathcal{A} + \mathcal{B}F)x$  is strangeness-free. Let us assume for simplicity that this preliminary feedback has already been performed, so that  $\mathcal{E}\dot{x} = \mathcal{A}x$  is strangeness-free, i.e., the matrix

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}$$

is square and invertible. Numerical methods to compute such preliminary feedbacks are discussed in detail in [6, 5, 7].

From (5) we can read off that an initial condition is *consistent* if  $A_2x(t_0) + B_2u(t_0) = 0$  and thus the possible input functions  $u$  are restricted by this constraint. We then define the two subspaces

$$\mathbb{E}_d := \text{Im}(\mathcal{E}^T), \quad \mathbb{E}_a := \ker(\mathcal{E}),$$

where  $\text{Im}$  and  $\ker$  denote the *image* and *kernel*, respectively. Using the singular value decomposition, see [19],

$$U^T \mathcal{E} V = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix},$$

with orthogonal  $U, V$  and nonsingular diagonal  $\Sigma \in \mathbb{R}^{d,d}$  and the *Moore-Penrose inverse* of  $\mathcal{E}$  given by

$$\mathcal{E}^+ = V^T \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U \in \mathbb{R}^{n,m},$$

we can numerically compute four projectors

$$(6a) \quad P_d := \mathcal{E}^+ \mathcal{E} = V^T \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} V,$$

$$(6b) \quad P_a := I - \mathcal{E}^+ \mathcal{E} = V^T \begin{bmatrix} 0 & 0 \\ 0 & I_{n-d} \end{bmatrix} V,$$

$$(6c) \quad Q_d := \mathcal{E} \mathcal{E}^+ = U^T \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} U,$$

$$(6d) \quad Q_a := I - \mathcal{E} \mathcal{E}^+ = U^T \begin{bmatrix} 0 & 0 \\ 0 & I_{n-d} \end{bmatrix} U.$$

Then  $P_d, P_a$  are orthogonal projectors onto the subspaces  $\mathbb{E}_d, \mathbb{E}_a$ , respectively.

The computation of these projectors can be implemented in backward stable methods, the only subtlety is the usual decision which singular values are set to 0. The analysis for this numerical rank decision is well understood, see [16, 17]. Using the projectors, we can then partition the state additively as  $x = x_d + x_a$  with the two parts

$$(7) \quad x_d := P_d x, \quad x_a := P_a x$$

and have the following result.

**THEOREM 5.** [3] *Let the projectors  $P_d, P_a$  and  $Q_d, Q_a$  be defined as in (6) and the variables  $x_d$  and  $x_a$  as in (7). Then,  $x = x_d + x_a$  is a solution of (2) if and only if  $x_d$  and  $x_a$  are solutions of the system*

$$(8a) \quad \dot{x}_d = G_d x_d + B_d u,$$

$$(8b) \quad x_a = G_a x_d + B_a u,$$

where

$$(9a) \quad G_a := -(Q_a A P_a)^+ (Q_a A P_d), \quad G_d := \mathcal{E}^+ A (P_d + G_a),$$

$$(9b) \quad B_a := -(Q_a A P_a)^+ \mathcal{B}, \quad B_d := \mathcal{E}^+ \mathcal{B} + \mathcal{E}^+ A B_a,$$

respectively. Moreover, an initial value  $x_0$  is consistent if and only if it satisfies

$$(P_a - G_a)x_0 = B_a u(t_0)$$

at the initial time  $t_0$ .

The solution set of system (8) can be used to compute the set of state reachable points of the first order system (2) from a consistent initial condition  $x(t_0) = x_0$ . This is discussed in the next section.

**3. The reachable set in the first order case.** Following the previous analysis, a reachable point  $x_1$  of (2) can be determined by computing its differential component  $x_{d_1}$  and then the algebraic component  $x_{a_1}$  via the relations in (8). If  $x_0$  is consistent, then  $x_{d_0} = P_d x_0$  is an initial condition for the standard ordinary differential equation system  $\dot{x}_d = G_d x_d + B_d u$  and the solution is

$$(10) \quad x_d = e^{G_d(t-t_0)} x_{d_0} + \int_{t_0}^t e^{G_d(t-\tau)} B_d u(\tau) d\tau.$$

The resulting algebraic component  $x_a$  is given by

$$(11) \quad x_a = G_a \left[ e^{G_d(t-t_0)} x_{d_0} + \int_{t_0}^t e^{G_d(t-\tau)} B_d u(\tau) d\tau \right] + B_a u.$$

This shows that the state responses  $x_d$  and  $x_a$  are uniquely determined by the initial condition  $x_0$ , the control input  $u(\tau)$  for  $t_0 \leq \tau \leq t$  and the initial time  $t_0$ . If the initial time is  $t_0 = 0$  and the initial condition is  $x_0 = 0$ , then we have  $x_d(0) = P_d x_0 = 0$  and hence, the solution (10) is given by

$$(12) \quad x_d = \int_0^t e^{G_d(t-\tau)} B_d u(\tau) d\tau.$$

With the operator

$$\mathcal{L}_d(u, t) := \int_0^t e^{G_d(t-\tau)} B_d u(\tau) d\tau,$$

the set of reachable points  $x_{d_1}$  from  $x(0) = 0$  is the image space of  $\mathcal{L}_d(u, t)$  which, see [13], is the image space of the symmetric positive semidefinite *Gramian* matrix

$$(13) \quad W(p, t) := \int_0^t p(\tau)^2 e^{G_d(t-\tau)} B_d B_d^T e^{G_d^T(t-\tau)} d\tau,$$

where  $p(\tau)$  is a polynomial which is not identically zero. Moreover, by defining the subspace

$$(14) \quad \mathcal{X}_d := \text{Im}(\mathcal{C}_d),$$

where  $\mathcal{C}_d = [B_d \ G_d B_d \ \cdots \ G_d^{n-1} B_d]$  is the *controllability matrix* of the dynamical part, it follows as in [12] that

$$(15) \quad \text{Im}[W(p, t)] = \mathcal{X}_d.$$

The algebraic part  $x_a = G_a x_d + B_a u$  imposes restrictions in the reachability set. Thus to compute a reachable point  $x_1$ , assume that  $x_{d_1}$  is a reachable point due to the state response  $x_d$  for some input  $u_1$  at a finite time  $t_1 > 0$ . Then at  $t_1$ , the algebraic component  $x_{a_1}$  is given by  $x_{a_1} = G_a x_{d_1} + B_a u_1(t_1)$ . Since  $x_{d_1} \in \mathcal{X}_d$ , we can write  $x_{d_1} = \mathcal{C}_d w$  for some  $w \in \mathbb{R}^{nm}$ . Hence,  $x_{a_1}$  takes the form

$$x_{a_1} = G_a \mathcal{C}_d w + B_a c,$$

where  $c = u_1(t_1) \in \mathbb{R}^m$ . Defining the set

$$(16) \quad \begin{aligned} \mathcal{X}_a &:= \{x_{a_1} \in \mathbb{R}^n \mid x_{a_1} = G_a \mathcal{C}_d w + B_a c, c = u(t_1)\} \\ &= \text{Im}[G_a \mathcal{C}_d \ B_a], \end{aligned}$$

we have the complete characterization of the set  $\mathcal{R}_0$  of reachable points of system (2) from an initial condition  $x_0 = 0$  at  $t_0 = 0$ , given by

$$(17) \quad \mathcal{R}_0 = \mathcal{X}_d + \mathcal{X}_a = \{x_{d_1} + x_{a_1} \mid x_{d_1} \in \mathcal{X}_d, x_{a_1} \in \mathcal{X}_a\}.$$

see [13, 14] for details. Since  $\mathcal{X}_d = \text{Im}(\mathcal{C}_d)$  and  $\mathcal{X}_a = \text{Im}[G_a \mathcal{C}_d \ B_a]$ , the set  $\mathcal{R}_0$  can be computed in a numerically backward stable way using the matrices defined in (9). Moreover,  $\dim(\mathcal{R}_0) = \dim(\mathcal{X}_d) + \dim(\mathcal{X}_a)$ , where  $\dim$  refers to the dimension of a subspace.

When we start from an arbitrary consistent initial condition  $x(t_0) = x_0$ , then by performing a time shift and considering  $x - x_0$  we obtain as reachable set the affine space  $x_0 + \mathcal{R}_0$ . A first order example illustrating the computational process is presented in [13, 14]. In the following section we use these results to compute the set of reachable points for a second order descriptor system in trimmed first order form.

**4. Reachability set for second order descriptor systems.** To extend the procedure to compute the reachable set to second order descriptor systems of the form (1), we first reformulate the strangeness-free second order system by a derivative array approach directly from the behavior form of the second order system, see [29, 35],

$$(18) \quad \mathbf{M}\ddot{z}(t) + \mathbf{D}\dot{z}(t) = \mathbf{K}z(t),$$

where  $\mathbf{M} := [M \ 0] \in \mathbb{R}^{n, n+m}$ ,  $\mathbf{D} := [D \ 0] \in \mathbb{R}^{n, n+m}$ ,  $\mathbf{K} := [-K \ B] \in \mathbb{R}^{n, n+m}$ , by introducing the new variable  $z = [x^T \ u^T]^T$ . Then, the derivative array takes the form

$$(19) \quad \mathcal{M}_\mu \ddot{z}_\mu + \mathcal{L}_\mu \dot{z}_\mu = \mathcal{N}_\mu z_\mu,$$

where  $z_\mu = [z \ \dot{z} \ \ddot{z} \ \dots \ z^{(\mu)}]^T$ ,

$$\begin{aligned} \mathcal{M}_\mu &= \begin{bmatrix} \mathbf{M} & 0 & 0 & \dots & 0 \\ \mathbf{D} & \mathbf{M} & 0 & \dots & 0 \\ -\mathbf{K} & \mathbf{D} & \mathbf{M} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -\mathbf{K} & \mathbf{D} & \mathbf{M} \end{bmatrix}, \\ \mathcal{L}_\mu &= \begin{bmatrix} \mathbf{D} & 0 & \dots & 0 \\ -\mathbf{K} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad \mathcal{N}_\mu = \begin{bmatrix} \mathbf{K} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}. \end{aligned}$$

The strangeness-free system is obtained by computing projection matrices  $Z_0$ ,  $Z_1$ ,  $Z_2$  and  $Z_3$ , see [35], such that

$$(20) \quad \begin{bmatrix} \widehat{M}_1 \\ 0 \\ 0 \end{bmatrix} \ddot{z} + \begin{bmatrix} \widehat{D}_1 \\ \widehat{D}_2 \\ 0 \end{bmatrix} \dot{z} = \begin{bmatrix} \widehat{K}_1 \\ \widehat{K}_2 \\ \widehat{K}_3 \end{bmatrix} z,$$

where

$$\begin{aligned} \widehat{M}_1 &= Z_0^T \mathbf{M} \in \mathbb{R}^{d_1, n+m}, & \widehat{D}_1 &= Z_0^T \mathbf{D} \in \mathbb{R}^{d_1, n+m}, \\ \widehat{K}_1 &= Z_0^T \mathbf{K} \in \mathbb{R}^{d_1, n+m}, & \widehat{D}_2 &= Z_1^T Z_2^T \mathcal{L}_\mu \begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix} \in \mathbb{R}^{d_2, n+m}, \\ \widehat{K}_2 &= Z_1^T Z_2^T \mathcal{N}_\mu \begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix} \in \mathbb{R}^{d_2, n+m}, \\ \widehat{K}_3 &= Z_3^T \mathcal{N}_\mu \begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix} \in \mathbb{R}^{a, n+m}. \end{aligned}$$

In the strangeness-free system (20), the matrices  $\widehat{M}_1$ ,  $\widehat{D}_2$  and  $\widehat{K}_3$  are of full row rank and, furthermore by our assumption that the system is regular, also the matrix

$$(21) \quad \widehat{H} := \begin{bmatrix} \widehat{M}_1 \\ \widehat{D}_2 \\ \widehat{K}_3 \end{bmatrix} \in \mathbb{R}^{n, n+m}$$

is of full row rank  $n$ . Moreover, (20) can be represented (in the original variables) as

$$(22a) \quad M_1 \ddot{x} + D_1 \dot{x} + K_1 x = B_1 u,$$

$$(22b) \quad D_2 \dot{x} + K_2 x = B_2 u,$$

$$(22c) \quad K_3 x = B_3 u,$$

where  $M_i$ ,  $D_i$  and  $K_i$ ,  $B_i$  are obtained from  $\widehat{M}_i$ ,  $\widehat{D}_i$  and  $\widehat{K}_i$ , respectively, by inserting the partitioned vector  $z$  and taking all state terms to the left.

As in the first order case, system (22) may, however, not be strangeness-free as a free system with  $u = 0$ , since the matrix

$$H := \begin{bmatrix} M_1 \\ D_2 \\ K_3 \end{bmatrix} \in \mathbb{R}^{n, n}$$

may not be of full row rank  $n$ . But since the behavior system is strangeness-free, again there exists a feedback  $u = -G\dot{x} - Fx + \tilde{u}$  such that in the closed loop system

$$(23a) \quad M_1 \ddot{x} + (D_1 + B_1 G) \dot{x} + (K_1 + B_1 F) x = B_1 \tilde{u},$$

$$(23b) \quad (D_2 + B_2 G) \dot{x} + (K_2 + B_2 F) x = B_2 \tilde{u},$$

$$(23c) \quad (K_3 + B_3 F) x = B_3 \tilde{u},$$

the matrix

$$H_F := \begin{bmatrix} M_1 \\ D_2 + B_2 G \\ K_3 + B_3 F \end{bmatrix} \in \mathbb{R}^{n, n}$$



is nonsingular, see [24].

In order to apply the results from the first order case, we transform system (23) into a trimmed first-order formulation that does not need unnecessary smoothness requirements. For this, we determine an orthogonal matrix  $W$  (this can be done in a numerically stable way via a singular value or QR decomposition with pivoting [19]) such that in

$$(24) \quad H_F W = \begin{bmatrix} \tilde{M}_{11} & 0 & 0 \\ \tilde{D}_{21} & \tilde{D}_{22} & 0 \\ \tilde{K}_{31} & \tilde{K}_{32} & \tilde{K}_{33} \end{bmatrix}$$

the matrices  $\tilde{M}_{11} \in \mathbb{R}^{d_1, d_1}$ ,  $\tilde{D}_{22} \in \mathbb{R}^{d_2, d_2}$ ,  $\tilde{K}_{33} \in \mathbb{R}^{a, a}$  are square and nonsingular.

Then, by performing a change of basis, i.e., by setting  $x = Ww$ , we obtain  $M_1 W = [\tilde{M}_{11} \ 0 \ 0]$ ,  $(D_1 + B_1 G)W = [\tilde{D}_{11} \ \tilde{D}_{12} \ \tilde{D}_{13}]$ ,  $(D_2 + B_2 G)W = [\tilde{D}_{21} \ \tilde{D}_{22} \ 0]$  and  $(K_j + B_j F)W = [\tilde{K}_{j1} \ \tilde{K}_{j2} \ \tilde{K}_{j3}]$ ,  $j = 1, 2, 3$ . In this setting, system (23) can be represented as

$$\begin{bmatrix} \tilde{M}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \\ \ddot{w}_3 \end{bmatrix} + \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} \\ \tilde{D}_{21} & \tilde{D}_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} + \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} & \tilde{K}_{13} \\ \tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23} \\ \tilde{K}_{31} & \tilde{K}_{32} & \tilde{K}_{33} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \tilde{u}.$$

By introducing a new variable  $v = \dot{w}_1$  we obtain the strangeness-free first order system

$$(25) \quad \begin{bmatrix} \tilde{M}_{11} & \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} \\ 0 & \tilde{D}_{21} & \tilde{D}_{22} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I_{d_1} & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} + \begin{bmatrix} 0 & \tilde{K}_{11} & \tilde{K}_{12} & \tilde{K}_{13} \\ 0 & \tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23} \\ 0 & \tilde{K}_{31} & \tilde{K}_{32} & \tilde{K}_{33} \\ -I_{d_1} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ 0 \end{bmatrix} \tilde{u}.$$

Note that  $v$  can be interpreted as the *velocity vector* of the dynamic part.

This formulation is nice but it is not expressed in the original variables, so that the physical meaning of the variables has changed. To avoid this difficulty, by inserting  $w = W^T x$  in terms of the original variables, one obtains

$$\begin{bmatrix} \tilde{M}_{11} & (D_1 + B_1 G) \\ 0 & (D_2 + B_2 G) \\ 0 & 0 \\ 0 & W_1^T \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 & K_1 + B_1 F \\ 0 & K_2 + B_2 F \\ 0 & K_3 + B_3 F \\ -I_{d_1} & 0 \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ 0 \end{bmatrix} \tilde{u},$$

where  $W_1^T = [I_{d_1} \ 0 \ 0] W^T$ . Reordering the block rows we obtain

$$\begin{bmatrix} \tilde{M}_{11} & (D_1 + B_1 G) \\ 0 & (D_2 + B_2 G) \\ 0 & W_1^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 & K_1 + B_1 F \\ 0 & K_2 + B_2 F \\ -I_{d_1} & 0 \\ 0 & K_3 + B_3 F \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ B_3 \end{bmatrix} \tilde{u}.$$

This system is strangeness-free as a free system with  $\tilde{u} = 0$  and we write this as

$$(26) \quad L\dot{\xi} = S\xi + \tilde{B}\tilde{u},$$

where

$$L = \begin{bmatrix} \tilde{M}_{11} & (D_1 + B_1G) \\ 0 & (D_2 + B_2G) \\ 0 & W_1^T \\ 0 & 0 \end{bmatrix}, \quad S = - \begin{bmatrix} 0 & K_1 + B_1F \\ 0 & K_2 + B_2F \\ -I_{d_1} & 0 \\ 0 & K_3 + B_3F \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ B_3 \end{bmatrix}, \quad \xi = \begin{bmatrix} v \\ x \end{bmatrix}, \quad v = W_1^T \dot{x}.$$

Since  $v(t) = W_1^T \dot{x}(t)$ , it is clear that the components of a reachable velocity vector  $v(t)$  will lie on hyperplanes defined by  $W_1^T \dot{x}(t)$  for  $t \geq 0$ . Recall that in Example 3 we have discussed that the velocity components of a constrained mechanical system are restricted in some directions and as a result the velocity reachable points are also. The velocity directions that one can reach are defined by the relation  $v(t) = W_1^T \dot{x}(t)$  and the other components of the velocity vector  $v \in \mathbb{R}^n$  will remain unchanged. After this restriction of the velocity vector, we can apply the results of the first order case and obtain

$$(27a) \quad \dot{\xi}_d(t) = G_d \xi_d(t) + B_d \tilde{u}(t),$$

$$(27b) \quad \xi_a(t) = G_a \xi_d(t) + B_a \tilde{u}(t),$$

where

$$(28a) \quad G_a := -(Q_a S P_a)^+ Q_a S P_d, \quad G_d := L^+ S (P_d + G_a),$$

$$(28b) \quad B_a := -(Q_a S P_a)^+ \tilde{B}, \quad B_d := L^+ \tilde{B} + L^+ S B_a,$$

with projectors

$$(29) \quad P_d = L^+ L, \quad P_a = I - L^+ L, \quad Q_d = L L^+ = \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_a = I - L L^+.$$

In (27) the extended state vector  $\xi$  satisfies  $\xi = \xi_d + \xi_a$  where  $\xi_d = [v_d^T, x_d^T]^T$  and  $\xi_a = [v_a^T, x_a^T]^T$ . It follows that an initial value

$$\xi_0 = \begin{bmatrix} W_1^T \dot{x}(t_0) \\ x(t_0) \end{bmatrix}$$

is consistent if and only if it satisfies

$$(P_a - G_a) \xi_0 = B_a \tilde{u}(t_0)$$

at time  $t_0$ . Hence the reachable points for a given control input  $\tilde{u}$  can be determined from the relations in (27).

Let us first discuss the question of computing the reachable position (for the dynamic part) and the reachable velocity points from zero initial conditions, i.e.,  $x(0) = x_0 = 0$  and  $v(0) = v_0 = 0$ . Corresponding to these initial conditions the solution of (27a) is given by

$$\xi_d = \int_0^t e^{G_d(t-\tau)} B_d \tilde{u}(\tau) d\tau,$$

and hence,

$$(30a) \quad v_d = [I_{d_1} \quad 0] \left( \int_0^t e^{G_d(t-\tau)} B_d \tilde{u}(\tau) d\tau \right),$$

$$(30b) \quad x_d = [0 \quad I_n] \left( \int_0^t e^{G_d(t-\tau)} B_d \tilde{u}(\tau) d\tau \right).$$

If we partition the matrix  $S$  as

$$S = \left[ \begin{array}{c|ccc} 0 & K_{11} & K_{12} & K_{13} \\ 0 & K_{21} & K_{22} & K_{23} \\ I_{d_1} & 0 & 0 & 0 \\ 0 & K_{31} & K_{32} & K_{33} \end{array} \right],$$

and partition the projectors  $Q_a$  and  $P_a$  accordingly, then we obtain the following consistency conditions from the algebraic relation (27b):

$$(31a) \quad v_a = 0,$$

$$(31b) \quad x_a = [0 \quad I_n] \left[ G_a \left( \int_0^t e^{G_d(t-\tau)} B_d \tilde{u}(\tau) d\tau \right) + B_a \tilde{u} \right].$$

Hence the reachable position and velocity points are determined by (30) and (31). Furthermore, since  $v_a = 0$ , the reachable vectors  $v = v_d$  are determined by (30a). We thus have proved the following reachability conditions.

**THEOREM 6.** *Consider a second order system in the regularized first order form (25).*

1. *A state vector  $x_1 := x_{d_1} + x_{a_1}$  is reachable from a consistent initial condition  $x(t_0) = x_0$  if there exists a control input  $\tilde{u}(t)$ , and a finite time  $t_1 > 0$  such that  $x_d(t_1) = x_{d_1}$  and  $x_a(t_1) = x_{a_1}$ .*
2. *An extended state vector  $\xi_1 := \xi_{d_1} + \xi_{a_1}$  is reachable from a consistent initial condition  $\xi(t_0) = \xi_0$  if there exists a control input  $\tilde{u}(t)$  and a finite time  $t_1 > 0$  such that  $\xi(t_1) = \xi_1$ .*

We can also give explicit formulas for the reachability sets and the associated controls. The reachable set  $\mathcal{R}_{\xi_0}$  associated with the trimmed first order system (25) is given by

$$(32) \quad \mathcal{R}_{\xi_0} = \mathcal{W}_d + \mathcal{W}_a,$$

where

$$\begin{aligned} \mathcal{W}_d &:= \text{Im}(\mathcal{C}_d), \text{ with } \mathcal{C}_d = [B_d \quad G_d B_d \quad \cdots \quad G_d^{n-1} B_d], \\ \mathcal{W}_a &:= \text{Im}([G_a \mathcal{C}_d \quad B_a]). \end{aligned}$$

Hence, the set of reachable extended state vectors  $\xi_1$  is contained in the subspace  $\mathcal{R}_{\xi_0}$ . Since the system (26) is strangeness-free, the control input is given by

$$(33) \quad \tilde{u} = B_d^T e^{G_d^T(t-\tau)} k, \quad 0 \leq \tau \leq t,$$

where  $k \in \mathbb{R}^{n+d_1}$  satisfies  $W(t)k = \xi_{d_1}$  ( $W(t)$  is the associated Gramian matrix). With the subspaces

$$\begin{aligned} \mathcal{X}_d &:= \text{Im}([0 \quad I_n] \mathcal{C}_d), \\ \mathcal{X}_a &:= \text{Im}([0 \quad I_n] [G_a \mathcal{C}_d \quad B_a]), \end{aligned}$$

the set of reachable position vectors  $x_1$  is contained in the subspace

$$(34) \quad \mathcal{R}_{x_0} = \mathcal{X}_d + \mathcal{X}_a.$$

Note that the derivative array approach and thus also the construction of the reachable sets can be extended in a similar way for systems of order higher than two.

REMARK 7. We have discussed at several places that the strangeness-free behavior model may not be strangeness-free for  $u(t) = 0$ . This situation can be circumvented by using feedback controls of the form  $u(t) = Fx(t) + \tilde{u}(t)$  and  $u(t) = -G\dot{x}(t) + Fx(t) + \tilde{u}(t)$  for first and second order descriptor systems, respectively. Recall that while computing the reachable space for the first order descriptor system we have used the strangeness-free behavior model and the control input  $u(t)$  instead of  $\tilde{u}(t)$ . Hence, we have considered a smooth control input  $u(t)$ . On the other hand, by using a preliminary feedback control, we do not need smoothness conditions on the input function  $\tilde{u}(t)$ . This situation is demonstrated in the second order descriptor system, where the control input  $\tilde{u}(t)$  takes the form  $\tilde{u}(t) = B_d^T e^{G_d^T(t-\tau)}k$ ,  $0 \leq \tau \leq t$  for first as well as second order descriptor systems.

**5. Numerical Procedure and Example.** We have demonstrated how to obtain the reachability sets of second order systems of the form (1) in a numerically stable way in the previous section. We have implemented this construction in the following numerically backward stable procedure. The first step is a procedure for the construction of the strangeness-free form (20), adapted from [35]. From this form we can then explicitly construct the reachability sets.

ALGORITHM 8. *Construction of regularized second order descriptor system.*

**Input:** Matrices  $\mathbf{M} := [M \ 0]$ ,  $\mathbf{D} := [D \ 0]$ ,  $\mathbf{K} := [-K \ B]$  constructed from the second order descriptor system (1).

**Output:** The projection matrices  $Z_0$ ,  $Z_1$ ,  $Z_2$  and  $Z_3$  that are required to obtain the strangeness-free system (20).

**Steps:**

1. Start with  $\mu = 0$ .
2. Form the derivative array according to (19).
3. Compute 'a' from the relation:  $\text{rank}([\mathcal{M}_\mu \ \mathcal{L}_\mu]) = (\mu + 1)n - a$ .
4. Compute  $Z$  such that  $Z^T \mathcal{M}_\mu = 0$  and  $Z_3$  such that  $Z_3^T [\mathcal{M}_\mu \ \mathcal{L}_\mu] = 0$ .
5. Let  $R_z$  be a basis matrix for the range space of  $Z^T Z_3$ . Then, compute  $Z_2 = ZR_z$ .
6. Compute  $T_3$  such that  $Z_3^T \mathcal{N}_\mu [I \ 0 \ \dots \ 0]^T T_3 = 0$ .
7. Compute the rank of  $Z_2^T \mathcal{L}_\mu [I \ 0 \ \dots \ 0]^T T_3$  and denote it as  $d_2$ .
8. Compute the singular value decomposition (SVD) of  $Z_2^T \mathcal{L}_\mu [I \ 0 \ \dots \ 0]^T$  and write it as

$$Z_2^T \mathcal{L}_\mu [I \ 0 \ \dots \ 0]^T = U_{z_2} S_{z_2} V_{z_2}^T.$$

Then, form  $Z_1$  by taking the first  $d_2$  columns of  $U_{z_2}$ .

9. Compute  $T_2$  such that  $Z_1^T Z_2^T \mathcal{L}_\mu [I \ 0 \ \dots \ 0]^T T_2 = 0$ .
10. Compute the SVD of  $\mathbf{M} T_3 T_2 = U_m S_m V_m^T$ . Denote  $d_1$  as the rank of  $S_m$ . Then form  $Z_0$  by taking the first  $d_1$  columns of  $U_m$ .
11. Check if  $a + d_2 + d_1 = n$  then **stop**, otherwise increase the value of  $\mu$  by one and follow the procedure from Step 2.

The only critical point in this procedure is the decision about the numerical rank. However, as has been shown in [26], and improved in [21], if there is doubt about the

rank decision in the construction of the strangeness-free form one just goes one level further to obtain a better rank decision.

ALGORITHM 9. *Computation of reachable sets.*

**Input:** The matrices  $M_1, D_1, D_2, K_i$  and  $B_i$  for  $i = 1, \dots, 3$  associated with the strangeness-free system (22), the feedback gain matrices  $F$  and  $G$  to obtain (23).

**Output:** The matrices  $W_d, W_a, X_d$  and  $X_a$  that are required to compute the reachable subspaces.

**Steps:**

1. Construct the matrix  $H_F := \begin{bmatrix} M_1 \\ D_2 + B_2G \\ K_3 + B_3F \end{bmatrix}$ .
2. Compute the orthogonal matrix  $W$  (by performing either singular value or QR decomposition with pivoting) such that (24) holds.
3. Compute the matrices  $L, S, \bar{B}$  as in (26), and the projectors  $P_d, P_a, Q_d, Q_a$  as in (29).
4. Compute the matrices  $G_d, G_a, B_d$  and  $B_a$  as in (28).
5. Compute  $C_d = [B_d \ G_d B_d \ \dots \ G_d^{n-1} B_d]$  and  $C_a := [G_a C_d \ B_a]$ .
6. Compute the SVD of  $C_d$  and write it as  $C_d = U_{cd} S_{cd} V_{cd}^T$ . Then construct a matrix  $W_d$  by taking the columns of  $U_{cd}$  corresponding to the non-zero diagonal elements of  $S_{cd}$ . Similarly, write  $C_a = U_{ca} S_{ca} V_{ca}^T$  by computing the SVD of  $C_a$ , and then construct a matrix  $W_a$  by taking the columns of  $U_{ca}$  corresponding to the non-zero diagonal elements of  $S_{ca}$ .
7. Compute the SVD of  $[0 \ I_n] C_d$  and write it as  $[0 \ I_n] C_d = U_{xd} S_{xd} V_{xd}^T$ . Then construct a matrix  $X_d$  by taking the columns of  $U_{xd}$  corresponding to the non-zero diagonal elements of  $S_{xd}$ . Similarly, write  $[0 \ I_n] C_a = U_{xa} S_{xa} V_{xa}^T$  by computing the SVD of  $[0 \ I_n] C_a$ , and then construct a matrix  $X_a$  by taking the columns of  $U_{xa}$  corresponding to the non-zero diagonal elements of  $S_{xa}$ .

The subspaces  $\mathcal{W}_d$  and  $\mathcal{W}_a$  are spanned by the columns of the matrices  $W_d$  and  $W_a$ , respectively. Similarly,  $\mathcal{X}_d$  and  $\mathcal{X}_a$  are spanned by the columns of  $X_d$  and  $X_a$ , respectively. Then the reachable subspaces are as in (32) and (34).

To illustrate the technique for the computation of the reachable set, we consider the model of the three-link mobile manipulator from [20], see Example 3, for which the concrete data can be found in [14]. Carrying out the regularization procedure, it follows that this system has strangeness-index  $\mu = 2$  in the behavior setting, and we obtain a strangeness-free behavior model (20), where

$$\begin{aligned} \widehat{M}_1 &= [-14.5135 \ 42.3643 \ -38.6090 \ 0 \ 0 \ 0 \ 0 \ 0], \\ \widehat{D}_1 &= [4.7430 \ 4.8373 \ -4.8373 \ 0 \ 0 \ 0 \ 0 \ 0], \\ \widehat{K}_1 &= [-83.9703 \ 10.6511 \ -54.9365 \ -0.1875 \\ &\quad -0.6330 \ 0.6752 \ -1.1064 \ -0.2019], \\ \widehat{D}_2 &= 10^{-13} [-0.2297 \ 0.0275 \ -0.7357 \ -0.0000 \ -0.0090 \\ &\quad 0.0016 \ -0.0029 \ -0.0077], \\ \widehat{K}_2 &= [0.6477 \ 0.2356 \ -0.8514 \ -0.0034 \ 0.0048 \\ &\quad -0.0000 \ 0.0025 \ 0.0023], \end{aligned}$$

$$\widehat{K}_3 = \begin{bmatrix} 0.3913 & 0.2676 & -0.8869 & -0.0038 & 0.0054 \\ 4.2024 & 2.3386 & 1.0103 & -0.0334 & 0.0473 \\ 4.8339 & 3.7396 & 1.6016 & -0.0534 & 0.0757 \\ -0.0000 & 0.0028 & 0.0026 \\ -0.0000 & 0.0246 & 0.0227 \\ -0.0000 & 0.0394 & 0.0363 \end{bmatrix}.$$

The matrix  $\widehat{H}$  in (21) is of full row rank 5, when setting the tolerance equal to the machine precision *eps* in computing the rank. The resulting system is also strangeness-free for  $u = 0$ , since the rank of  $H$  is 5. However, since the matrix  $H$  is nearly singular, we may improve the robustness of the system by using a preliminary feedback control  $u = -G\dot{x} - Fx + \tilde{u}$ , with

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \end{bmatrix}, \quad F = 0,$$

which leads to an  $H_F$  which is robustly of full row rank.

To obtain a first order formulation we choose the following orthogonal matrix

$$W = \begin{bmatrix} -0.2455 & 0.0804 & 0.7585 & 0.5982 & -0.0100 \\ 0.7165 & -0.2347 & -0.2289 & 0.6158 & -0.0026 \\ -0.6530 & -0.2877 & -0.5363 & 0.4508 & 0.0009 \\ 0 & -0.9250 & 0.2908 & -0.2444 & -0.0005 \\ 0 & 0.0000 & 0.0076 & 0.0071 & 0.9999 \end{bmatrix}$$

which is obtained by computing a  $QR$  decomposition of  $H_F^T$ . Then following the discussion in Section 4 we obtain an associated first order model  $L\dot{\xi} = S\xi + \tilde{B}u$ , where

$$L = \begin{bmatrix} 59.1273 & 4.7430 & 4.8373 & -10.3692 & -2.0190 & 0 \\ 0 & -0.0000 & 0.0000 & 0.0124 & 0.0229 & -0.0000 \\ 0 & -0.2455 & 0.7165 & -0.6530 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} 0 & -83.9703 & 10.6511 & -54.9365 & -0.1875 & -0.6330 \\ 0 & 0.6477 & 0.2356 & -0.8514 & -0.0034 & 0.0048 \\ 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.3913 & 0.2676 & -0.8869 & -0.0038 & 0.0054 \\ 0 & 4.2024 & 2.3386 & 1.0103 & -0.0334 & 0.0473 \\ 0 & 4.8339 & 3.7396 & 1.6016 & -0.0534 & 0.0757 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} 0.6752 & -1.1064 & -0.2019 \\ -0.0000 & 0.0025 & 0.0023 \\ 0 & 0 & 0 \\ -0.0000 & 0.0028 & 0.0026 \\ -0.0000 & 0.0246 & 0.0227 \\ -0.0000 & 0.0394 & 0.0363 \end{bmatrix}.$$

The subspaces  $\mathcal{W}_d$  and  $\mathcal{W}_a$  corresponding to the trimmed first order formulation of

the second order system are spanned by the columns of

$$W_d = \begin{bmatrix} -0.6642 & 0.7365 \\ -0.2664 & -0.0903 \\ 0.5678 & 0.4962 \\ -0.3576 & -0.3963 \\ 0.1939 & 0.2149 \\ -0.0000 & -0.0000 \end{bmatrix}, \quad W_a = \begin{bmatrix} -0.0001 & 0.0145 & 0.1274 \\ -0.0041 & 0.3219 & -0.9040 \\ -0.0078 & 0.6566 & 0.0169 \\ -0.0070 & 0.5994 & 0.3584 \\ 0.0038 & -0.3250 & -0.1943 \\ 0.9999 & 0.0119 & -0.0002 \end{bmatrix}$$

and the subspaces  $\mathcal{X}_d$  and  $\mathcal{X}_a$  are spanned by the columns of

$$X_d = \begin{bmatrix} -0.2953 & -0.8571 \\ 0.7572 & 0.0594 \\ -0.5122 & 0.4498 \\ 0.2777 & -0.2439 \\ -0.0000 & 0.0000 \end{bmatrix}, \quad X_a = \begin{bmatrix} -0.0041 & -0.3223 & 0.9107 \\ -0.0078 & -0.6566 & -0.0186 \\ -0.0070 & -0.5993 & -0.3627 \\ 0.0038 & 0.3250 & 0.1967 \\ 0.9999 & -0.0119 & 0.0002 \end{bmatrix},$$

respectively. Hence, the reachable spaces are  $\mathcal{R}_{\xi_0} = \mathcal{W}_d + \mathcal{W}_a$  and  $\mathcal{R}_{x_0} = \mathcal{X}_d + \mathcal{X}_a$ .

**6. Conclusion.** We have presented a procedure to compute the reachable set for second order linear time-invariant descriptor systems. We have shown how to obtain a strangeness-free behavior model corresponding to the original model via a derivative array approach. The strangeness-free system is then decoupled into a differential part and an algebraic part with the help of a projection method. The coefficient matrices of the projected system are used to define two subspaces and the addition of these two subspaces gives the reachable space, from the origin, of the original descriptor system. To apply this technique for second order systems, the constructed strangeness-free behavior model is transformed into a first order system by a trimmed first order formulation, which is then used to compute the reachable vectors.

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