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Well-posedness, Internal Stability and Input-output Stability in Networked Multi-agent Systems

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***Abstract*—In this article, we consider a feedback interconnection, which consists of a networked multi-agent system with its distributed controller and two sets of external input signals. Assuming that the agents are identical, and the dynamics of each agent and its local controller are represented by transfer functions, the well-posedness condition, internal stability and input-output stability of the feedback interconnection are investigated. A controller is designed by solving a Diophantine equation, and a gain for the network is selected using root-locus of a unity feedback system to achieve input-output stability. The developed results are demonstrated with a numerical example.**

***Index Terms*—Multi-agent system, Distributed control, Linear time invariant system, Stabilization.**

I. Introduction

A NETWORKED multi-agent system (NMAS) consists of a group of dynamical systems (*agents*) and a communication network among them, so as to exchange information. Such systems can suitably be represented by a graph, by assigning the agents as *vertices* and communication links as *edges*. This graphical representation then helps to use the available results on *algebraic graph theory* for the analysis and synthesis of control strategies. In particular, the *Laplacian matrix* and its spectrum play an important role in achieving control objectives, such as consensus ([1]–[3]), synchronization ([4]–[7]) and formation ([8]–[11]). The above control objectives are achieved by using a *distributed control architecture* for NMAS, where each agent has its own local controller, which is responsible to produce appropriate control signal for the agent, by taking the (relative) information (state and/or output) from neighborhood agents. The control methodologies are initially developed for agents, whose dynamics are of single and double integrator type [1], [2], [12], [13], and later they are considered to be of homogeneous higher order systems [4]–[6]. Although a majority of the work on NMAS is in the state-space (autonomous) framework, some of the available literature use transfer function (input-output) set-up for system analysis and control design. For instance, in [8], [9], the agents and controller dynamics are represented by transfer functions, and the problem of *formation stabilization* is investigated using Nyquist stability criterion. In addition, the system theoretic properties, such as controllability [14], stability margins (gain and phase) [14], influence of transmission zeros [15], [16] and compensator design for consensus and stabilization [14], [17] are investigated in the transfer function representation setting.

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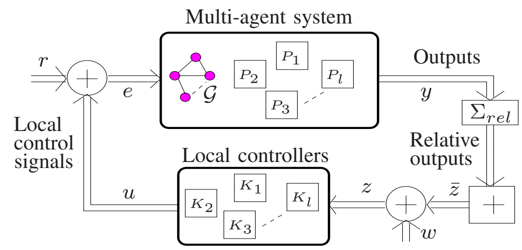

Fig. 1. Feedback interconnection $\boldsymbol{\Sigma}$, where $\boldsymbol{r}$ and $\boldsymbol{w}$ are external signals, and $\boldsymbol{e}$, $\boldsymbol{y}$, $\boldsymbol{z}$ and $\boldsymbol{u}$ are internal signals. The block $\boldsymbol{\Sigma_{rel}}$ takes the outputs of neighborhood agents and produces relative outputs.

Motivated by the seminal work [8], in this article, we consider the following problem for NMAS. Suppose that a distributed control architecture is implemented in the NMAS, where the dynamic behaviour of each agent and its local controller are described by transfer functions. Assume that the feedback configuration is exposed to two sets of external signals. One set of external signals is added with the sum of relative outputs (with respect to neighborhood agents), that are available for inputs to the local controllers, and another set is added with the inputs to the agents, as depicted in Fig. 1. This feedback configuration, denoted as Σ, resembles with the feedback interconnection of two transfer functions. Some of the fundamental topics: *well-posedness*, *internal stability* and *synthesis of a stabilizing controller*, for feedback interconnection of two transfer functions, are well studied in the literature (see [18], [19] and the references therein). In this article, we investigate these problems for the feedback interconnection Σ (FI-Σ), where we assume that the agents in the NMAS are identical, and hence, they have same transfer functions, and the underlying communication network is bidirectional, that is, the representative graph is undirected.

The contributions of this work are as follows. We propose a necessary and sufficient condition for FI-Σ such that it is physically realizable, that is, the interconnection is well-posed. Then, by representing the closed loop system in state space form, it is proved using two specially defined gain matrices that internal stability and input-output stability of FI-Σ are equivalent, provided that state space realizations (SSRs) of agent and its local controller are stabilizable and detectable. Further, we show that input-output stability of FI-Σ can be

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inferred by verifying the stability of a family of polynomials. To achieve input-output stability, a synthesis procedure is proposed, where in the first step, the controller is obtained by solving a Diophantine equation [20], and then, *a gain* for the network is fixed with the help of *root-locus* of a unity feedback system.

Notations: I_n: an identity matrix of size $n \times n$, $\mathbf{0}$: matrix with zero entries, $\otimes$: Kronecker product, $\mathbb{C}$: complex plane, $\mathbb{C}^-$: open left half (excluding imaginary axis) of $\mathbb{C}$, $\mathbb{C}^+$: open right half (excluding imaginary axis) of $\mathbb{C}$, diag$\{\bullet\}$: diagonal matrix and blkdiag$\{\bullet\}$: block diagonal matrix.

II. Feedback interconnection set-up

Consider a NMAS, which consists of a group of l identical agents. The dynamics of each agent is represented by transfer function: $P(s) = \frac{\beta(s)}{\alpha(s)}$. It is assumed that $P(s)$ is proper, that is, $\lim_{s\to\infty} P(s) = d_p \in \mathbb{R}$. By defining a set $\mathcal{N} := \{1, 2, \cdots, l\}$, the input-output relation for each agent is:

$$y_i(s) = P(s)e_i(s), \quad \forall\ i \in \mathcal{N}, \tag{1}$$

where $e_i(s)$ is the Laplace transform of (time domain) signal $e_i(t)$, which is input to the i^{th} agent, and $y_i(s)$ is the Laplace transform of $y_i(t)$, which is output of the i^{th} agent. Assume that each agent can establish communication with its neighbors through appropriate communication links, and the nature of communication is bidirectional. With this assumption, the NMAS is represented by an undirected (connected) graph $\mathcal{G}$ [2], where i^{th} vertex v_i corresponds to the i^{th} agent, and an undirected edge between the two vertices v_i and v_j with weight $a_{ij} > 0$ corresponds to the communication link between agents: i and j, for $i, j \in \mathcal{N}$. We say that agent j is *neighbor* of agent i, if v_i and v_j are *adjacent* in $\mathcal{G}$, that is, there exists an edge between v_i and v_j. The neighborhood set of i^{th} agent is defined as: $N(i) := \{j \neq i \mid v_i \text{ and } v_j \text{ are adjacent in } \mathcal{G}\}$. The $(i,j)^{th}$ element of an *adjacency matrix* $\mathbf{A}$ of $\mathcal{G}$ is a_{ij}, if v_i and v_j are adjacent, otherwise it is zero. It is assumed that there are no self loops in $\mathcal{G}$, and hence, $a_{ii} = 0$. In addition, we assume that $a_{ij} = a_{ji}$. The degree of i^{th} vertex $d(i)$ and the *degree matrix* $\mathbf{\Delta}$ of $\mathcal{G}$ are defined as: $d(i) := \sum_{j\in N(i)} a_{ij}$ and $\mathbf{\Delta} := \text{diag}\{d(1), d(2), \cdots, d(l)\}$, respectively. The *Laplacian matrix* $\mathbf{L}$ of $\mathcal{G}$ is $\mathbf{L} = \mathbf{\Delta} - \mathbf{A}$.

Assume that the local controllers for the agents are identical, and their dynamics are represented by the transfer function: $K(s) = \frac{\psi(s)}{\phi(s)}$. Let $K(s)$ be proper, that is, $\lim_{s\to\infty} K(s) = d_k \in \mathbb{R}$. The input-output relation for each controller is:

$$u_i(s) = K(s)z_i(s), \quad \forall\ i \in \mathcal{N}, \tag{2}$$

where $z_i(s)$, which is input to the i^{th} controller, and $u_i(s)$, which is output from the i^{th} controller, are the Laplace transforms of $z_i(t)$ and $u_i(t)$, respectively. Since agent i can communicate with its neighbors, its local controller $K(s)$ has access to the outputs of agent i and its neighbors. Hence, for $t \geq 0$ and $i \in \mathcal{N}$, the *neighborhood relative output* information available for i^{th} controller is defined as:

$$\bar{z}_i(t) := -\gamma \left[\sum_{j\in N(i)} a_{ij}(y_j(t) - y_i(t)) \right] - y_i(t), \tag{3}$$

where $N(i)$ is the neighborhood set of agent i, The quantity γ, referred to as *network gain* (gain for the network), is a small positive real number satisfying: $0 < \gamma < 1$. The exact choice of γ will become clear in Section V. Defining the vectors: $\bar{\mathbf{z}} := [\bar{z}_1\ \bar{z}_2\ \cdots\ \bar{z}_l]^T$ and $\mathbf{y} := [y_1\ y_2\ \cdots\ y_l]^T$, (3) can compactly be written as: $\bar{\mathbf{z}} = (\gamma\mathbf{L} - \mathbf{I}_l)\mathbf{y}$, where $\mathbf{L}$ is the Laplacian matrix of $\mathcal{G}$. Let us now define the signals $e_i(t)$ and $z_i(t)$ as follows (for $t \geq 0$):

$$e_i(t) := u_i(t) + r_i(t), \ z_i(t) := \bar{z}_i(t) + w_i(t), \ \forall i \in \mathcal{N}, \tag{4}$$

where $\bar{z}_i(t)$ is as in (3), and $r_i(t)$ and $w_i(t)$ are two external signals, which may be considered as added input (reference or disturbance) and measurement noise, respectively. Observe that the combination of (1), (2), (3) and (4) represents a feedback interconnection (closed loop system) with two sets of external input signals. This feedback interconnection is referred to FI-Σ, which was introduced in Section I.

Following vectors are defined, which are used in remaining parts of this article: $\mathbf{r} := [r_1\ r_2\ \cdots\ r_l]^T$, $\mathbf{w} := [w_1\ w_2\ \cdots\ w_l]^T$, $\mathbf{e} := [e_1\ e_2\ \cdots\ e_l]^T$, $\mathbf{z} := [z_1\ z_2\ \cdots\ z_l]^T$, $\mathbf{u} := [u_1\ u_2\ \cdots\ u_l]^T$, $\mathbf{r}_{cl} := [\mathbf{r}^T\ \mathbf{w}^T]^T$ and $\mathbf{y}_{cl} := [\mathbf{y}^T\ \mathbf{u}^T\ \mathbf{e}^T\ \mathbf{z}^T]^T$. For notational simplicity, the input arguments: s (frequency domain) and t (time domain) are omitted from the signals.

III. Well-posedness of FI-Σ

In this section, we derive a condition such that FI-Σ is *well-posed*, (one may refer to Chapter 5 of [18], [19] for more details on the feedback interconnection of two transfer functions). For this, we use the following definition.

Definition 1: The FI-Σ is well-posed, if all of the transfer function matrices, from external input signals: $\mathbf{r}$ and $\mathbf{w}$ to the internal output signals: $\mathbf{y}$, $\mathbf{u}$, $\mathbf{e}$ and $\mathbf{z}$, are proper.

Let $\mathbf{L_g} := (\mathbf{I}_l - \gamma\mathbf{L}) \in \mathbb{R}^{l\times l}$. Further, define the following matrices of size $l \times l$: $\mathbf{D_p}(\mathbf{s}) := P(s)\mathbf{I}_l$, $\mathbf{D_k}(\mathbf{s}) := K(s)\mathbf{I}_l$, $\mathbf{D_{pk}}(\mathbf{s}) := P(s)K(s)\mathbf{I}_l$, $\mathbf{L_p}(\mathbf{s}) = -P(s)\mathbf{L_g}$ and $\mathbf{L_{pk}}(\mathbf{s}) = -P(s)K(s)\mathbf{L_g}$. Then, relation (1), using (2) and (4), can compactly be written as:

$$\mathbf{y} = \mathbf{D_{pk}}(\mathbf{s})\mathbf{z} + \mathbf{D_p}(\mathbf{s})\mathbf{r} = \mathbf{D_{pk}}(\mathbf{s})(\bar{\mathbf{z}} + \mathbf{w}) + \mathbf{D_p}(\mathbf{s})\mathbf{r}. \tag{5}$$

Recall that $\bar{\mathbf{z}} = (\gamma\mathbf{L} - \mathbf{I}_l)\mathbf{y}$. Hence, relation (5) can be written as: $\mathbf{y} = \mathbf{D_{pk}}(\mathbf{s})\left(-\mathbf{L_g}\mathbf{y} + \mathbf{w}\right) + \mathbf{D_p}(\mathbf{s})\mathbf{r}$, which then becomes: $\mathbf{G}(\mathbf{s})\mathbf{y} = [\mathbf{D_p}(\mathbf{s})\ \ \mathbf{D_{pk}}(\mathbf{s})]\,\mathbf{r}_{cl}$, where $\mathbf{G}(\mathbf{s}) = \mathbf{I}_l - \mathbf{L_{pk}}(\mathbf{s})$. Similarly, following relations are also obtained: $\mathbf{G}(\mathbf{s})\mathbf{u} = [\mathbf{L_{pk}}(\mathbf{s})\ \ \mathbf{D_k}(\mathbf{s})]\mathbf{r}_{cl}$, $\mathbf{G}(\mathbf{s})\mathbf{e} = [\mathbf{I}_l\ \ \mathbf{D_k}(\mathbf{s})]\mathbf{r}_{cl}$, $\mathbf{G}(\mathbf{s})\mathbf{z} = [\mathbf{L_p}(\mathbf{s})\ \ \mathbf{I}_l]\mathbf{r}_{cl}$. Hence, the closed loop transfer function matrix $\mathbf{H}(\mathbf{s})$ of size $4l \times 2l$, from input $\mathbf{r}_{cl}$ to the output $\mathbf{y}_{cl}$, is

$$\mathbf{H}(\mathbf{s}) = \left(\widehat{\mathbf{G}}(\mathbf{s})\right)^{-1} \begin{bmatrix} \mathbf{D_p}(\mathbf{s}) & \mathbf{L_{pk}}(\mathbf{s}) & \mathbf{I}_l & \mathbf{L_p}(\mathbf{s}) \\ \mathbf{D_{pk}}(\mathbf{s}) & \mathbf{D_k}(\mathbf{s}) & \mathbf{D_k}(\mathbf{s}) & \mathbf{I}_l \end{bmatrix}^T, \tag{6}$$

where $\widehat{\mathbf{G}}(\mathbf{s}) = \text{blkdiag}\{\mathbf{G}(\mathbf{s}), \mathbf{G}(\mathbf{s}), \mathbf{G}(\mathbf{s}), \mathbf{G}(\mathbf{s})\}$. It is then clear that $\mathbf{H}(\mathbf{s})$ is proper if and only if $\mathbf{G}(\mathbf{s})$ is invertible and its inverse is a proper transfer function matrix. Note that $\mathbf{G}(\mathbf{s})$ is proper, since $\mathbf{G}_\infty := \lim_{s\to\infty} \mathbf{G}(\mathbf{s}) = \mathbf{I}_l + d_p d_k \mathbf{L_g} \in \mathbb{R}^{l\times l}$. Hence, it follows from [20, Chapter 7, Corollary 3.13] that $\mathbf{G}^{-1}(s)$ exists and it is proper if and only if $\mathbf{G}_\infty$ is invertible. This leads to the following result.

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Lemma 1: The FI-Σ is well-posed if and only if the matrix $\mathbf{G}_\infty = \mathbf{I}_l + d_p d_k \mathbf{L_g}$ is invertible. In addition, FI-Σ is well-posed if one of the following statements holds: i) either $P(s)$ or $K(s)$ is strictly proper ($d_p = 0$ or $d_k = 0$), ii) $P(s)$ and $K(s)$ both are strictly proper ($d_p = d_k = 0$).

IV. STABILITY ANALYSIS OF FI-Σ

To investigate the internal stability of FI-Σ, we represent agent and its local controller in the state space form. Let

$$\dot{\mathbf{x}}_i = \mathbf{A_p}\mathbf{x}_i + \mathbf{b_p} e_i, \quad y_i = \mathbf{c_p}\mathbf{x}_i + d_p e_i, \quad \forall\, i \in \mathcal{N}, \qquad (7)$$

be the SSR of i^{th} agent whose transfer function is $P(s)$. In (7), $\mathbf{A_p} \in \mathbb{R}^{n_p \times n_p}$, $\mathbf{b_p} \in \mathbb{R}^{n_p}$, $\mathbf{c_p}^T \in \mathbb{R}^{n_p}$ and $d_p \in \mathbb{R}$ with initial state $\mathbf{x}_i(0) \in \mathbb{R}^{n_p}$. Let

$$\dot{\bar{\mathbf{x}}}_i = \mathbf{A_k}\bar{\mathbf{x}}_i + \mathbf{b_k} z_i, \quad u_i = \mathbf{c_k}\bar{\mathbf{x}}_i + d_k z_i, \quad \forall\, i \in \mathcal{N}, \qquad (8)$$

be the SSR of i^{th} controller whose transfer function is $K(s)$. In (8), $\mathbf{A_k} \in \mathbb{R}^{n_k \times n_k}$, $\mathbf{b_k} \in \mathbb{R}^{n_k}$, $\mathbf{c_k}^T \in \mathbb{R}^{n_k}$ and $d_k \in \mathbb{R}$ with initial state $\bar{\mathbf{x}}_i(0) \in \mathbb{R}^{n_k}$. Define $\mathbf{x} \in \mathbb{R}^{ln_p}$ and $\bar{\mathbf{x}} \in \mathbb{R}^{ln_k}$ as: $\mathbf{x} := [\mathbf{x}_1^T \ \mathbf{x}_2^T \ \cdots \ \mathbf{x}_l^T]^T$ and $\bar{\mathbf{x}} := [\bar{\mathbf{x}}_1^T \ \bar{\mathbf{x}}_2^T \ \cdots \ \bar{\mathbf{x}}_l^T]^T$, respectively. By writing $\mathbf{x_{cl}} = [\mathbf{x}^T \ \bar{\mathbf{x}}^T]^T$ as the state vector of overall closed loop system, the internal stability is defined as follows.

Definition 2: The autonomous form ($\mathbf{r}(t) = 0$ and $\mathbf{w}(t) = 0$, for $t \geq 0$) of FI-Σ, when it is well-posed, is internally stable if the state trajectory $\mathbf{x_{cl}}(t)$ converges asymptotically to the origin (zero) from every initial state $\mathbf{x_{cl}}(0) \in \mathbb{R}^{l(n_p+n_k)}$.

Define the following matrices: $\mathbf{L_{\bar{d}c}} := -\mathbf{L_g} \otimes d_k \mathbf{c_p}$, $\mathbf{D_{\bar{c}}} := \mathbf{I}_l \otimes \mathbf{c_k}$, $\mathbf{L_c} := -\mathbf{L_g} \otimes \mathbf{c_p}$, $\mathbf{L_{d\bar{c}}} := -\mathbf{L_g} \otimes d_p \mathbf{c_k}$, $\widetilde{\mathbf{G}}_\infty :=$ blkdiag$\{\mathbf{G}_\infty, \mathbf{G}_\infty\}$, $\widehat{\mathbf{G}}_\infty :=$ blkdiag$\{\mathbf{G}_\infty, \mathbf{G}_\infty, \mathbf{G}_\infty, \mathbf{G}_\infty\}$, $\mathbf{A_{cl}} := \begin{bmatrix} \mathbf{I}_l \otimes \mathbf{A_p} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_l \otimes \mathbf{A_k} \end{bmatrix} + \begin{bmatrix} \mathbf{I}_l \otimes \mathbf{b_p} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_l \otimes \mathbf{b_k} \end{bmatrix} \widetilde{\mathbf{G}}_\infty^{-1} \begin{bmatrix} \mathbf{L_{\bar{d}c}} & \mathbf{D_{\bar{c}}} \\ \mathbf{L_c} & \mathbf{L_{d\bar{c}}} \end{bmatrix}$, $\mathbf{B_{cl}} := \begin{bmatrix} \mathbf{I}_l \otimes \mathbf{b_p} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_l \otimes \mathbf{b_k} \end{bmatrix} \widetilde{\mathbf{G}}_\infty^{-1} \begin{bmatrix} \mathbf{I}_l & d_k \mathbf{I}_l \\ -d_p \mathbf{L_g} & \mathbf{I}_l \end{bmatrix}$. In addition, $\mathbf{C_{cl}} := \widehat{\mathbf{G}}_\infty^{-1} \begin{bmatrix} \mathbf{I}_l \otimes \mathbf{c_p} & \mathbf{I}_l \otimes d_p \mathbf{c_k} \\ \mathbf{L_{\bar{d}c}} & \mathbf{D_{\bar{c}}} \\ \mathbf{L_{\bar{d}c}} & \mathbf{D_{\bar{c}}} \\ \mathbf{L_c} & \mathbf{L_{d\bar{c}}} \end{bmatrix}$ and $\mathbf{D_{cl}} := \widehat{\mathbf{G}}_\infty^{-1} \begin{bmatrix} d_p \mathbf{I}_l & d_p d_k \mathbf{I}_l \\ -d_p d_k \mathbf{L_g} & d_k \mathbf{I}_l \\ \mathbf{I}_l & d_k \mathbf{I}_l \\ -d_p \mathbf{L_g} & \mathbf{I}_l \end{bmatrix}$. Then, with the assumption that FI-Σ is well-posed, the SSR of FI-Σ, considering $\mathbf{r}_{cl}$ as input and $\mathbf{y}_{cl}$ as output, is represented as follows (derivation is omitted due to the page limit constraint):

$$\dot{\mathbf{x}}_{\mathbf{cl}} = \mathbf{A_{cl}}\mathbf{x_{cl}} + \mathbf{B_{cl}}\mathbf{r}_{cl}, \quad \mathbf{y}_{cl} = \mathbf{C_{cl}}\mathbf{x_{cl}} + \mathbf{D_{cl}}\mathbf{r}_{cl}. \qquad (9)$$

Then, the transfer function matrix $\mathbf{H(s)}$ in (6) can also be obtained as follows: $\mathbf{H(s)} = \mathbf{C_{cl}}\left(sI - \mathbf{A_{cl}}\right)^{-1}\mathbf{B_{cl}} + \mathbf{D_{cl}}$. From (9), the state equation of the autonomous form of FI-Σ can be represented as: $\dot{\mathbf{x}}_{cl} = \mathbf{A_{cl}}\mathbf{x_{cl}}$. Hence, the following result holds, which follows from linear system theory [20].

Lemma 2: The autonomous form of FI-Σ, when it is well-posed, is internally stable if and only if the eigenvalues of matrix $\mathbf{A_{cl}}$ are located in $\mathbb{C}^-$, that is, $\mathbf{A_{cl}}$ is stable.

We now establish a relationship between *internal stability* and *input-output* stability of FI-Σ, where the following definition is used for input-output stability.

Definition 3: The FI-Σ, when it is well-posed, is input-output stable, if for a given set of bounded input signals: $\mathbf{r}(t)$ and $\mathbf{w}(t)$, the output signals: $\mathbf{y}(t)$, $\mathbf{u}(t)$, $\mathbf{e}(t)$ and $\mathbf{z}(t)$ are also bounded for $t \geq 0$.

Theorem 1: Let (7) and (8) be stabilizable and detectable realizations of $P(s)$ and $K(s)$, respectively. Let (9) be the SSR of FI-Σ. Then, FI-Σ is well-posed and internally stable if and only if $\mathbf{H(s)}$, as in (6), is proper and stable ($\mathbf{H(s)}$ is *stable* if all of its poles are in $\mathbb{C}^-$).

Proof: Assume that FI-Σ is well-posed and internally stable. Then, it follows from Lemma 2 that $\mathbf{A_{cl}}$ is stable. Since the poles of $\mathbf{H(s)}$ are subset of the eigenvalues of $\mathbf{A_{cl}}$ (there might be some uncontrollable and/or unobservable eigenvalues in $\mathbf{A_{cl}}$), $\mathbf{H(s)}$ is also stable. Further, since FI-Σ is well-posed, $\mathbf{G}_\infty$ is invertible, and hence, FI-Σ has SSR (9). Since $\lim_{s\to\infty} \mathbf{H(s)} = \mathbf{D_{cl}} \in \mathbb{R}^{4l \times 2l}$, $\mathbf{H(s)}$ is proper.

Now, assume that $\mathbf{H(s)}$ is proper and stable. Since $\mathbf{H(s)}$ is proper, $\mathbf{G(s)}$ is invertible and its inverse is a proper transfer function matrix. This implies $\mathbf{G}_\infty$ is invertible and FI-Σ is well-posed. Further, since $\mathbf{H(s)}$ is stable, its poles are in $\mathbb{C}^-$. The poles of $\mathbf{H(s)}$ are (both) controllable and observable eigenvalues of $\mathbf{A_{cl}}$ [20], and hence, they belong to $\mathbb{C}^-$. The other eigenvalues of $\mathbf{A_{cl}}$ are possibly uncontrollable and/or unobservable. We now prove that the uncontrollable and/or unobservable eigenvalues of $\mathbf{A_{cl}}$ are also contained in $\mathbb{C}^-$, by showing that (9) is both stabilizable and detectable.

Let $\boldsymbol{\Pi_b} := \begin{bmatrix} \mathbf{I}_l & d_k \mathbf{I}_l \\ -d_p \mathbf{L_g} & \mathbf{I}_l \end{bmatrix}$. Since $-d_p \mathbf{L_g}$ and $\mathbf{I}_l$ commute, $\det(\boldsymbol{\Pi}_b) = \det(\mathbf{I}_l + d_p d_k \mathbf{L_g}) = \det(\mathbf{G}_\infty)$ [21]. Hence, $\boldsymbol{\Pi}_b$ is invertible. For $\mathbf{f_p} \in \mathbb{R}^{1 \times n_p}$ and $\mathbf{f_k} \in \mathbb{R}^{1 \times n_k}$, define:

$$\mathbf{F_{cl}} := \boldsymbol{\Pi}_b^{-1} \widetilde{\mathbf{G}}_\infty \left(\begin{bmatrix} \mathbf{I}_l \otimes \mathbf{f_p} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_l \otimes \mathbf{f_k} \end{bmatrix} - \widetilde{\mathbf{G}}_\infty^{-1} \begin{bmatrix} \mathbf{L_{\bar{d}c}} & \mathbf{D_{\bar{c}}} \\ \mathbf{L_c} & \mathbf{L_{d\bar{c}}} \end{bmatrix} \right). \qquad (10)$$

Then, $\mathbf{A_{cl}} + \mathbf{B_{cl}}\mathbf{F_{cl}} = \begin{bmatrix} \mathbf{I}_l \otimes (\mathbf{A_p} + \mathbf{b_p f_p}) & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_l \otimes (\mathbf{A_k} + \mathbf{b_k f_k}) \end{bmatrix}$. Since it is assumed that $(\mathbf{A_p}, \mathbf{b_p})$ and $(\mathbf{A_k}, \mathbf{b_k})$ are stabilizable, by suitable choice of $\mathbf{f_p}$ and $\mathbf{f_k}$, one can make $(\mathbf{A_p} + \mathbf{b_p f_p})$ and $(\mathbf{A_k} + \mathbf{b_k f_k})$ stable. Hence, using $\mathbf{F_{cl}}$, the matrix $(\mathbf{A_{cl}} + \mathbf{B_{cl}}\mathbf{F_{cl}})$ can be made stable. It then follows from [18, Chapter 3] that the pair $(\mathbf{A_{cl}}, \mathbf{B_{cl}})$ is stabilizable. Next, it is shown that the pair $(\mathbf{A_{cl}}, \mathbf{C_{cl}})$ is also detectable. Let $\boldsymbol{\Pi_c} := \begin{bmatrix} \mathbf{I}_l & d_p \mathbf{I}_l \\ -d_k \mathbf{L_g} & \mathbf{I}_l \end{bmatrix}$. Then, $\begin{bmatrix} \mathbf{I}_l \otimes \mathbf{c_p} & \mathbf{I}_l \otimes d_p \mathbf{c_k} \\ \mathbf{L_{\bar{d}c}} & \mathbf{D_{\bar{c}}} \\ \mathbf{L_{\bar{d}c}} & \mathbf{D_{\bar{c}}} \\ \mathbf{L_c} & \mathbf{L_{d\bar{c}}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Pi_c} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{2l} \end{bmatrix} \begin{bmatrix} \mathbf{I}_l \otimes \mathbf{c_p} & \mathbf{0} \\ \mathbf{0} & \mathbf{D_{\bar{c}}} \\ \mathbf{L_{\bar{d}c}} & \mathbf{D_{\bar{c}}} \\ \mathbf{L_c} & \mathbf{L_{d\bar{c}}} \end{bmatrix}$. Since $-d_k \mathbf{L_g}$ and $\mathbf{I}_l$ in $\boldsymbol{\Pi_c}$ commute, it can be shown that the matrix $\begin{bmatrix} \boldsymbol{\Pi_c} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{2l} \end{bmatrix}$ is invertible. Define:

$$\begin{aligned} \boldsymbol{\Theta_{cl}} :=& \begin{bmatrix} \mathbf{I}_l \otimes \boldsymbol{\theta_p} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_l \otimes \boldsymbol{\theta_k} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Pi}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{2l} \end{bmatrix}^{-1} \widehat{\mathbf{G}}_\infty \\ &- \begin{bmatrix} \mathbf{I}_l \otimes \mathbf{b_p} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_l \otimes \mathbf{b_k} \end{bmatrix} \widetilde{\mathbf{G}}_\infty^{-1} \left[\mathbf{0} \ \mathbf{I}_{2l}\right] \widehat{\mathbf{G}}_\infty, \end{aligned} \qquad (11)$$

where $\boldsymbol{\theta_p} \in \mathbb{R}^{n_p}$ and $\boldsymbol{\theta_k} \in \mathbb{R}^{n_k}$. This choice of $\boldsymbol{\Theta_{cl}}$ yields: $\mathbf{A_{cl}} + \boldsymbol{\Theta}_{cl}\mathbf{C_{cl}} = \begin{bmatrix} \mathbf{I}_l \otimes (\mathbf{A_p} + \boldsymbol{\theta_p}\mathbf{c_p}) & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_l \otimes (\mathbf{A_k} + \boldsymbol{\theta_k}\mathbf{c_k}) \end{bmatrix}$. Since it is assumed that $(\mathbf{A_p}, \mathbf{c_p})$ and $(\mathbf{A_k}, \mathbf{c_k})$ are detectable, by suitable choice of $\boldsymbol{\theta_p}$ and $\boldsymbol{\theta_k}$, the matrices $(\mathbf{A_p} + \boldsymbol{\theta_p}\mathbf{c_p})$ and $(\mathbf{A_k} + \boldsymbol{\theta_k}\mathbf{c_k})$ can be made stable. This implies, $\boldsymbol{\Theta}_{cl}$ can be used to make $(\mathbf{A_{cl}} + \boldsymbol{\Theta}_{cl}\mathbf{C_{cl}})$ stable, and hence, the pair $(\mathbf{A_{cl}}, \mathbf{C_{cl}})$ is detectable. This leads to the conclusion that (9) is both stabilizable and detectable. Hence, all the eigenvalues of $\mathbf{A_{cl}}$ are in $\mathbb{C}^-$, and FI-Σ is internally stable. ■

It follows from linear system theory [20, Chapter 6] that FI-Σ is input-output stable if and only if $\mathbf{H(s)}$ is stable. Hence, according to Theorem 1, internal stability and input-output

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stability of FI-Σ are equivalent, provided that the SSRs of $P(s)$ and $K(s)$ are both stabilizable and detectable. Since the dynamics of agents and their local controllers are represented by transfer functions, we focus on input-output stability of FI-Σ, and then, internal stability can be inferred by using stabilizable and detectable SSRs of $P(s)$ and $K(s)$.

Theorem 2: Let $\{\lambda_1, \lambda_2, \cdots, \lambda_l\}$ be the eigenvalues of $\mathbf{L_g}$. For given $P(s)$ and $K(s)$, define $M_i(s) := 1 + \lambda_i P(s)K(s)$, for $i \in \mathcal{N}$. Then, transfer function matrix $\mathbf{H(s)}$ is stable if and only if the following set of transfer functions are stable:

$$\frac{1}{M_i(s)}, \; \frac{P(s)}{M_i(s)}, \; \frac{K(s)}{M_i(s)}, \; \frac{P(s)K(s)}{M_i(s)}; \tag{12}$$

equivalently, the family of polynomials: $\sigma_i(s) := \alpha(s)\phi(s) + \lambda_i \beta(s)\psi(s)$ are stable, for all $i \in \mathcal{N}$.

Proof: Since $\mathbf{L_g}$ is a symmetric matrix, there exists an orthogonal matrix $\mathbf{U} \in \mathbb{R}^{l\times l}$ ($\mathbf{U}^T\mathbf{U} = \mathbf{I}_l$) such that $\mathbf{U}^T\mathbf{L_g}\mathbf{U} = \mathbf{D}_\Lambda$, where $\mathbf{D}_\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_l\}$. For a given set of bounded input signals $\mathbf{r}$ and $\mathbf{w}$, let us define a new set of bounded input signals as follows: $\tilde{\mathbf{r}} := \mathbf{U}^T\mathbf{r}$ and $\tilde{\mathbf{w}} := \mathbf{U}^T\mathbf{w}$. Similarly, corresponding to the output signals: $\mathbf{y}$, $\mathbf{u}$, $\mathbf{e}$ and $\mathbf{z}$, define a new set of output signals as follows: $\tilde{\mathbf{y}} := \mathbf{U}^T\mathbf{y}$, $\tilde{\mathbf{u}} := \mathbf{U}^T\mathbf{u}$, $\tilde{\mathbf{e}} := \mathbf{U}^T\mathbf{e}$ and $\tilde{\mathbf{z}} := \mathbf{U}^T\mathbf{z}$, which are bounded whenever $\mathbf{y}$, $\mathbf{u}$, $\mathbf{e}$ and $\mathbf{z}$ are bounded. Since $\mathbf{y}_{cl} = \mathbf{H(s)}\mathbf{r}_{cl}$, with the newly defined signals, the input-output relation of transformed system is:

$$\begin{bmatrix}\tilde{\mathbf{y}}^T & \tilde{\mathbf{u}}^T & \tilde{\mathbf{e}}^T & \tilde{\mathbf{z}}^T\end{bmatrix}^T = \mathbf{U}_d^T\mathbf{H(s)}\mathbf{V}_d\begin{bmatrix}\tilde{\mathbf{r}}^T & \tilde{\mathbf{w}}^T\end{bmatrix}^T, \tag{13}$$

where $\mathbf{U}_d = \mathrm{blkdiag}\{\mathbf{U},\mathbf{U},\mathbf{U},\mathbf{U}\}$ and $\mathbf{V}_d = \mathrm{blkdiag}\{\mathbf{U},\mathbf{U}\}$. Define a new transfer function matrix $\widetilde{\mathbf{H}}(\mathbf{s}) := \mathbf{U}_d^T\mathbf{H(s)}\mathbf{V}_d$ and write its block elements (each of size $l \times l$) as: $\widetilde{\mathbf{H}}_{\mathbf{km}}(\mathbf{s})$, for $k = 1, \cdots, 4$ and $m = 1, 2$. Then, it can be shown that $\widetilde{\mathbf{H}}_{\mathbf{km}}(\mathbf{s})$ are diagonal matrices, where the diagonal elements of $\widetilde{\mathbf{H}}_{\mathbf{11}}(\mathbf{s})$, $\widetilde{\mathbf{H}}_{\mathbf{12}}(\mathbf{s})$, $\widetilde{\mathbf{H}}_{\mathbf{21}}(\mathbf{s})$, $\widetilde{\mathbf{H}}_{\mathbf{22}}(\mathbf{s})$, $\widetilde{\mathbf{H}}_{\mathbf{31}}(\mathbf{s})$ and $\widetilde{\mathbf{H}}_{\mathbf{41}}(\mathbf{s})$ are: $\frac{P(s)}{M_i(s)}$, $\frac{P(s)K(s)}{M_i(s)}$, $-\frac{\lambda_i P(s)K(s)}{M_i(s)}$, $\frac{K(s)}{M_i(s)}$, $\frac{1}{M_i(s)}$ and $-\frac{\lambda_i P(s)}{M_i(s)}$, respectively, for $i \in \mathcal{N}$. The other elements: $\widetilde{\mathbf{H}}_{\mathbf{32}}(\mathbf{s}) = \widetilde{\mathbf{H}}_{\mathbf{22}}(\mathbf{s})$ and $\widetilde{\mathbf{H}}_{\mathbf{42}}(\mathbf{s}) = \widetilde{\mathbf{H}}_{\mathbf{31}}(\mathbf{s})$. Then, for $i \in \mathcal{N}$, following relations are obtained from (13): $\tilde{y}_i = \frac{P(s)}{M_i(s)}\tilde{r}_i + \frac{P(s)K(s)}{M_i(s)}\tilde{w}_i$, $\tilde{u}_i = -\frac{\lambda_i P(s)K(s)}{M_i(s)}\tilde{r}_i + \frac{K(s)}{M_i(s)}\tilde{w}_i$, $\tilde{e}_i = \frac{1}{M_i(s)}\tilde{r}_i + \frac{K(s)}{M_i(s)}\tilde{w}_i$ and $\tilde{z}_i = -\frac{\lambda_i P(s)}{M_i(s)}\tilde{r}_i + \frac{1}{M_i(s)}\tilde{w}_i$. From these relations, it follows that the output signals $\tilde{y}_i$, $\tilde{u}_i$, $\tilde{e}_i$ and $\tilde{z}_i$ are bounded, for bounded input signals $\tilde{r}_i$ and $\tilde{w}_i$, if and only if the transfer functions in (12) are stable (poles are in $\mathbb{C}^-$). Further, the transformed system (13) is input-output stable if and only if the transfer function matrix $\widetilde{\mathbf{H}}(\mathbf{s})$ is stable. Hence, stability of the transfer functions in (12) is equivalent to the stability of $\widetilde{\mathbf{H}}(\mathbf{s})$. Further, $\mathbf{H(s)}$ and $\widetilde{\mathbf{H}}(\mathbf{s})$ are equivalent, in the sense that they have same Smith-McMillan form [20]. Hence, $\mathbf{H(s)}$ is stable if and only if the transfer functions in (12) are stable. The stability of $\sigma_i(s)$'s directly follows by replacing $P(s) = \frac{\beta(s)}{\alpha(s)}$ and $K(s) = \frac{\psi(s)}{\phi(s)}$ in any one set of the transfer functions in (12). In fact, it can be verified that the roots of $\sigma_i(s)$, for $i \in \mathcal{N}$, are the poles of $\widetilde{\mathbf{H}}(\mathbf{s})$ (roots of the least common denominator polynomial of all non-zero $1, 2, \cdots, 2l$-order minors of $\widetilde{\mathbf{H}}(\mathbf{s})$ [20]), and hence, they are also the poles of $\mathbf{H(s)}$. ■

V. Synthesis procedure for input-output stability of FI-Σ

According to Theorem 2, stability of the transfer function matrix $\mathbf{H(s)}$ is equivalent to the stability of a family of transfer functions in (12) or polynomials $\sigma_i(s)$'s. Note that the eigenvalues λ_i's of $\mathbf{L_g}$ are dependent on the network gain γ, since $\mathbf{L_g} = \mathbf{I}_l - \gamma L$. In this section, we provide a synthesis procedure for controller $K(s)$ and network gain γ such that the polynomials $\sigma_i(s)$'s become stable. Let

$$P(s) = \frac{\beta(s)}{\alpha(s)} = \frac{\beta_{n-1}s^{n-1} + \ldots + \beta_1 s + \beta_0}{\alpha_n s^n + \cdots + \alpha_1 s + \alpha_0}, \tag{14}$$

with $\alpha_n \neq 0$. Assume that: i) $P(s)$ is strictly proper ($d_p = 0$) and ii) $\alpha(s)$ and $\beta(s)$ are co-prime. Corresponding to $\alpha(s)$ and $\beta(s)$ in (14), define the following matrix of size $2n \times 2n$:

$$\mathbf{R}_{\alpha\beta} := \left[\begin{array}{cccc|cccc} \alpha_0 & 0 & \cdots & 0 & \beta_0 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_0 & \cdots & 0 & \beta_1 & \beta_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_0 & \beta_{n-1} & \beta_{n-2} & \cdots & \beta_0 \\ \alpha_n & \alpha_{n-1} & \cdots & \alpha_1 & 0 & \beta_{n-1} & \cdots & \beta_1 \\ 0 & \alpha_n & \cdots & \alpha_2 & 0 & 0 & \cdots & \beta_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n & 0 & 0 & \cdots & 0 \end{array}\right] \tag{15}$$

Let the controller $K(s)$ be written as follows:

$$K(s) := \frac{\psi(s)}{\phi(s)} = \frac{\psi_{n-1}s^{n-1} + \cdots + \psi_1 s + \psi_0}{\phi_{n-1}s^{n-1} + \cdots + \phi_1 s + \phi_0}. \tag{16}$$

Corresponding to $\phi(s)$ and $\psi(s)$ in (16), define a vector: $\boldsymbol{\xi} := [\phi_0 \; \phi_1 \; \cdots \; \phi_{n-1} \mid \psi_0 \; \psi_1 \; \cdots \; \psi_{n-1}]^T \in \mathbb{R}^{2n}$. Further, for a given set of $(2n-1)$ complex numbers (including their conjugates) with negative real parts, construct a $(2n-1)^{th}$ degree polynomial: $\sigma(s) = \sigma_{2n-1}s^{2n-1} + \cdots + \sigma_1 s + \sigma_0$ and define: $\boldsymbol{\sigma} := [\sigma_0 \; \sigma_1 \; \cdots \; \sigma_{2n-1}]^T \in \mathbb{R}^{2n}$. Now, write the Diophantine equation [20]: $\alpha(s)\phi(s) + \beta(s)\psi(s) = \sigma(s)$, where $\psi(s)$ and $\phi(s)$ need to be determined for given $\alpha(s)$, $\beta(s)$ and $\sigma(s)$. By comparing the coefficients of both sides of the Diophantine equation, following relation is obtained:

$$\mathbf{R}_{\alpha\beta}\boldsymbol{\xi} = \boldsymbol{\sigma}. \tag{17}$$

Since $\alpha(s)$ and $\beta(s)$ are co-prime, $\mathbf{R}_{\alpha\beta}$ is non-singular [20]. Hence, for a given $\boldsymbol{\sigma}$, one can always compute $\boldsymbol{\xi}$ from (17). Further, it can be observed from (17) that $\alpha_n\phi_{n-1} = \sigma_{2n-1}$. Since $\sigma(s)$ is a $(2n-1)^{th}$ degree polynomial, $\sigma_{2n-1} \neq 0$, and hence, $\phi_{n-1} \neq 0$. This implies $K(s)$, obtained by solving (17), is proper. Since $\beta(s)$ and $\alpha(s)$ are co-prime, $P(s)$ does not have hidden unstable pole-zero cancellations. Hence, a stabilizable and detectable SSR can be obtained for $P(s)$. Further, the controller $K(s)$, designed by solving (17), is also free from hidden unstable pole-zero cancellations, which can be shown using the Bezout identity for proper and stable transfer functions [18]. Hence, a stabilizable and detectable SSR can also be obtained for $K(s)$.

Remark 1: Note that the strictly properness assumption on $P(s)$ helps to ensure that $K(s)$, designed by solving (17), is proper. However, one can relax this assumption, and consider $P(s)$ to be proper. In such case, there is a possibility that $K(s)$

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becomes improper ($\phi_{n-1} = 0$, $\psi_{n-1} \neq 0$) for a specific choice of stable $\sigma(s)$. Hence, to overcome this difficulty, following suggestion is proposed. Since we only need $\boldsymbol{\sigma}$ to be stable, a linear matrix inequality (LMI) feasibility problem can be formulated, to solve for $\boldsymbol{\xi}$, using the LMI given in [22, Lemma-1]. For this, one needs to choose a stable polynomial $\widetilde{\sigma}(s)$ with coefficient vector $\widetilde{\boldsymbol{\sigma}}$, replace $\boldsymbol{\sigma}$ with $\mathbf{R}_{\alpha\beta}\boldsymbol{\xi}$ in the LMI and impose an extra linear constraint: $\phi_{n-1} - 1 = 0$ (enforcing $\phi(s)$ to be monic). A feasible solution ensures that $K(s)$ is proper and $\boldsymbol{\sigma} = \mathbf{R}_{\alpha\beta}\boldsymbol{\xi}$ is stable.

Theorem 3: For a given $P(s)$ as in (14), and a given stable polynomial $\sigma(s)$, assume that $K(s)$ is obtained by solving (17). Let $\kappa_{\min} < 1$ be a positive real number such that for all real values of κ in the interval: $[\kappa_{\min}, 1]$, the branches of root-locus of the unity feedback system, shown in Fig. 2, belong to $\mathbb{C}^-$. Let $d_{\max} := \max_i d(i)$, where $d(i)$ is the degree of i^{th} vertex of $\mathcal{G}$. Then, for $\gamma \leq \frac{1-\kappa_{\min}}{2d_{\max}}$, the polynomials $\sigma_i(s)$'s, defined in Theorem 2, are all stable.

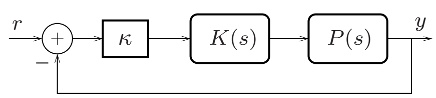

Fig. 2. Unity (negative) feedback system for root-locus drawing.

Proof: The root-locus of unity feedback system, shown in Fig. 2, is the locus of roots of the closed loop characteristic polynomial $\widehat{\sigma}(s) := \alpha(s)\phi(s) + \kappa\beta(s)\psi(s)$ for $\kappa \geq 0$ [23]. When $\kappa = 1$, $\widehat{\sigma}(s)$ becomes equal to the Diophantine equation, and hence, $\widehat{\sigma}(s) = \sigma(s)$. Since $K(s)$ is designed by solving (17) for a stable $\sigma(s)$, for $\kappa = 1$, the roots of $\widehat{\sigma}(s) = \sigma(s)$ are located in $\mathbb{C}^-$. Hence, the branches of root-locus lie in $\mathbb{C}^-$ for $\kappa = 1$. Since the root-locus is continuous in nature (with respect to κ), one can always find a positive number $\kappa_{min} < 1$ such that for all values of κ in the interval $[\kappa_{\min}, 1]$, the branches of root-locus lie in $\mathbb{C}^-$.

We now show that for $\gamma \leq \frac{1-\kappa_{\min}}{2d_{\max}}$, the polynomials $\sigma_i(s)$'s are stable. Corresponding to the matrix $\mathbf{L_g} = I - \gamma\mathbf{L}$, let us define a disc D_i in $\mathbb{C}$ as follows: $D_i := \{s \in \mathbb{C} \; : \; |s - c_i| \leq \rho_i\}$, where center $c_i = 1 - \gamma d(i)$ and radius $\rho_i = \gamma d(i)$. Then, according to the Geršgorin disc theorem [2], the eigenvalues: $\{\lambda_1, \lambda_2, \cdots, \lambda_l\}$ of $\mathbf{L_g}$ lie in $\bigcup_{i=1}^{l} D_i$. Since $\gamma \leq \frac{1-\kappa_{\min}}{2d_{\max}}$ and $\kappa_{\min} < 1$, we have: $2\gamma d_{\max} < 1$, which implies $\gamma d_{\max} < 0.5$. Further, by definition, $d(i) \leq d_{\max}$, and hence, for all $i \in \mathcal{N}$, $\gamma d(i) < 0.5$. It is then easy to notice that $\bigcup_{i=1}^{l} D_i = D_{\max}$, where the disc $D_{\max}$ has center at $1 - \gamma d_{\max}$ and radius $\gamma d_{\max}$, as illustrated in Fig. 3. Hence, the eigenvalues λ_i's of $\mathbf{L_g}$ lie in the disc $D_{\max}$. In fact, since $\mathbf{L_g}$ is a symmetric matrix, λ_i's are located only in the intersection of $D_{\max}$ and positive real axis of $\mathbb{C}$. Hence, for $i \in \mathcal{N}$, we have the following relation: $1 - 2\gamma d_{\max} \leq \lambda_i \leq 1$, that is, λ_i's belong to the interval: $[1 - 2\gamma d_{\max}, 1]$. Further, since $\gamma \leq \frac{1-\kappa_{\min}}{2d_{\max}}$, we have: $\kappa_{\min} \leq 1 - 2\gamma d_{\max}$. This leads to the following set inclusion relation: $[1 - 2\gamma d_{\max}, 1] \subseteq [\kappa_{\min}, 1]$. Recall that for $\kappa \in [\kappa_{\min}, 1]$, the branches of root-locus belong to $\mathbb{C}^-$. It is then clear, by comparing with the polynomial $\widehat{\sigma}(s) = \alpha(s)\phi(s) + \kappa\beta(s)\psi(s)$, that the roots of $\sigma_i(s)$ are located in $\mathbb{C}^-$, and hence, they are all stable. ■

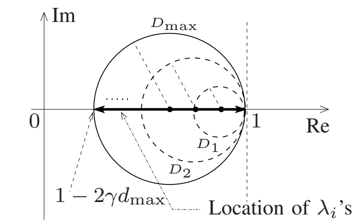

Fig. 3. Geršgorin discs in the complex plane for matrix $\mathbf{L_g}$.

Synthesis steps: We now summarize the synthesis steps for controller $K(s)$ and the network gain γ such that FI-Σ is input-output stable. Let $P(s)$ be as in (14). Then, choose a stable polynomial $\sigma(s)$ and compute $K(s)$ by solving (17) (if $P(s)$ is proper, then consider the procedure given in Remark 1 to compute $K(s)$). With the resulted $K(s)$, draw the root-locus of unity feedback system, shown in Fig. 2, and determine $\kappa_{\min}$. Choose γ such that $\gamma \leq \frac{1-\kappa_{\min}}{2d_{\max}}$. Then, according to Theorem 2 and Theorem 3, $\mathbf{H(s)}$ is stable, and hence, FI-Σ is input-output stable. Further, since $P(s)$ is strictly proper, according to Lemma 1, FI-Σ is well-posed (if $P(s)$ is proper, then it is important to ensure that $\mathbf{G}_\infty$ is invertible).

VI. Demonstrative Example

Example 1: Consider a NMAS with three agents and the inter-agent communication link is as per the graph, shown in Fig. 4. Let the weights on the communication links be: $a_{12} = 0.8$ and $a_{13} = 2$. Then, $\mathbf{L} = \begin{bmatrix} 2.8 & -0.8 & -2 \\ -0.8 & 0.8 & 0 \\ -2 & 0 & 2 \end{bmatrix}$ and $d_{\max} = 2.8$.

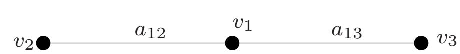

Fig. 4. The graph $\mathcal{G}$ of the NMAS with three agents.

Assume that the agents are identical with transfer function: $P(s) = \frac{s-1}{s^3-4s^2+s+6}$. We now follow the synthesis steps, as mentioned in Section V, to compute $K(s)$ and γ. To construct $\sigma(s)$, following set of complex numbers: $-2\pm1i$, $-1\pm0.5000i$ and -3 are chosen. Then, by solving (17) for $\boldsymbol{\xi}$, we obtain: $K(s) = \frac{54.75s^2+207.5s+152.2}{s^2+13s+28.5}$. Using $P(s)$ and $K(s)$, the root-locus of the unity feedback system is drawn, and is depicted in Fig. 5. It is observed that $\kappa_{min} = 0.8740$, and hence, $\frac{1-\kappa_{\min}}{2d_{\max}} = 0.0225$. By choosing $\gamma = 0.0225$, the eigenvalues of $\mathbf{L_g}$ are $\lambda_1 = 0.8978$, $\lambda_2 = 0.9762$ and $\lambda_3 = 1$. With the designed $K(s)$ and γ, the roots of $\sigma_1(s)$ are: -4.8758, -2.7610, $-0.1605 \pm 1.5557i$, -1.0421, $\sigma_2(s)$ are: -3.4874, -2.5886, $-0.8708 \pm 1.1564i$, -1.1825 and $\sigma_3(s)$ are: -3, -2 ± 1, $-1\pm0.5i$. Hence, $\sigma_i(s)$'s are stable. To verify internal stability, SSRs of $P(s)$ and $K(s)$ are obtained in MATLAB using *tf2ss* command. Then, the matrix $\mathbf{A_{cl}}$, as in (9), is computed, and it is observed that its eigenvalues are equal to the roots of $\sigma_i(s)$, for $i = 1, 2, 3$. To verify input-output stability, FI-Σ is simulated in MATLAB Simulink (version: R2017a), for which, three different bounded signals, as shown in Fig. 6 are considered as external signals $r_i(t)$, for $i = 1, 2, 3$. Similarly, a single bounded signal, as shown in Fig. 6 is considered as external signals $w_i(t)$, for $i = 1, 2, 3$. The closed loop system is simulated for 150 seconds. The output signals

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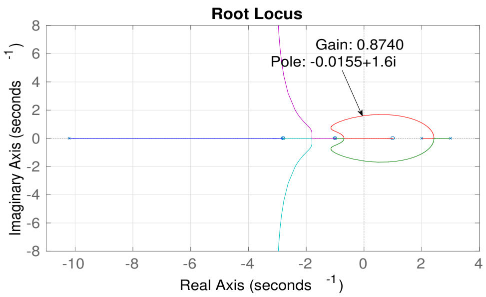

Fig. 5. Root-locus of unity feedback system for Example 1.

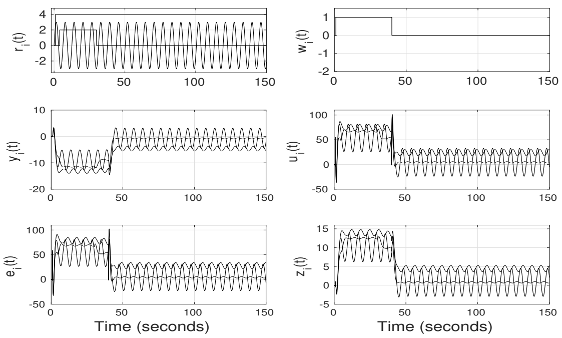

Fig. 6. Simulation results for FI-$\boldsymbol{\Sigma}$, where the input signals are $\boldsymbol{r_i(t)}$, $\boldsymbol{w_i(t)}$ and output signals are $\boldsymbol{y_i(t)}$, $\boldsymbol{u_i(t)}$, $\boldsymbol{e_i(t)}$ and $\boldsymbol{z_i(t)}$.

$y_i(t)$, $u_i(t)$, $e_i(t)$ and $z_i(t)$ are depicted in Fig. 6, which are also bounded. Hence, FI-Σ is input-output stable.

VII. Conclusion

Considering the NMAS with its distributed controller as a feedback interconnection, in this work, the problems of well-posedness, internal and input-out stability are investigated. A controller synthesis procedure and selection of a network gain are proposed to achieve input-output (also internal) stability. The advantages of this work are as follows. First, with the proposed feedback control architecture, there is no need of communication between the local controllers, since they produce appropriate control signals using only the outputs of their own and neighborhood agents. Second, the proposed approach can also be applied to a large size connected network (no limitation on network size), as long as the agents are of single-input single-output systems.

Although we have not investigated in this work on achieving collective motion of NMAS (consensus or synchronization), it is believed that this feedback set-up will serve as a unified platform to address the problems, such as: (i) formation stabilization and (ii) output consensus or synchronization, when the agents are subjected to step change in their inputs $r_i(t)$. Problem (ii) has possible applications in the areas, such as industrial conveyor belt system, railway traction system and wall cleaning robotic manipulator system (where a group of robotic manipulators are used to clean a long wall). For instance, in an industrial conveyor belt system, a group of (direct current) motors need to rotate in the same speed to collectively drive a long conveyor belt. Hence, when the motors face step change in their input voltages (possibly due to the availability of different magnitude of voltage sources at different locations or failure of some of the voltage sources), it is required, through appropriate control action that the speed of all motors synchronize to a certain value so as to collectively drive the belt. These problems, however, need further theoretical investigation, and hence, are left for future research. In addition, it would be interesting to extend the proposed approach for NMAS with multi-input multi-output agent dynamics.

References

[1] W. Ren and R. W. Beard, *Distributed Consensus in Multi-vehicle Cooperative Control (Communications and Control Engineering Series)*. London: Springer-Verlag, 2008.

[2] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods for Multiagent Networks*. Princeton, NJ: Princeton University Press, 2010.

[3] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 57, no. 1, pp. 213–224, 2010.

[4] H. Zhang, F. L. Lewis, and A. Das, "Optimal design for synchronization of cooperative systems: State feedback, observer and output feedback," *IEEE Transactions on Automatic Control*, vol. 56, no. 8, pp. 1948–1952, 2011.

[5] H. L. Trentelman, K. Takaba, and N. Monshizadeh, "Robust synchronization of uncertain linear multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 6, pp. 1511–1523, 2013.

[6] G. S. Seyboth, W. Ren, and F. Allgower, "Cooperative control of linear multi-agent systems via distributed output regulation and transient synchronization," *Automatica*, vol. 68, pp. 132–139, 2016.

[7] D. U. Patil and S. Datta, "Synchronization of multi-agent systems with distributed reduced norm state feedback control," in *Indian Control Conference (ICC)*, pp. 95–100, 2019.

[8] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1465–1476, 2004.

[9] Z. Jin and R. M. Murray, "Double-graph control strategy of multi-vehicle formations," in *IEEE Conference on Decision and Control*, vol. 2, pp. 1988–1994, 2004.

[10] K. K. Oh, M. C. Park, and H. S. Ahn, "A survey of multiagent formation control," *Automatica*, vol. 53, pp. 424–440, 2015.

[11] X. Sun and C. G. Cassandras, "Optimal dynamic formation control of multi-agent systems in constrained environments," *Automatica*, vol. 73, pp. 169–179, 2016.

[12] R. Olfati-Saber and R. M. Murray, "Distributed cooperative control of multiple vehicle formations using structural potential functions," *IFAC Proceedings Volumes*, vol. 35, no. 1, pp. 495–500, 2002.

[13] W. Ren, "On consensus algorithms for double-integrator dynamics," *IEEE Transactions on Automatic Control*, vol. 53, no. 6, pp. 1503–1509, 2008.

[14] M. Yoon and K. Tsumura, "Transfer function representation of cyclic consensus systems," *Automatica*, vol. 47, pp. 1974–1982, 2011.

[15] B. Briegel, D. Zelazo, M. Burger, and F. Allgower, "On the zeros of consensus networks," in 50^{th} *IEEE Conference on Decision and Control and European Control Conference*, pp. 1890–1895, 2011.

[16] I. Herman, D. Martinec, and M. Sebek, "Zeros of transfer functions in networked control with higher-order dynamics," in 19^{th} *IFAC World Congress*, pp. 9177–9182, 2014.

[17] A. Sakaguchi and T. Ushio, "Dynamic pinning consensus control of multi-agent systems," *IEEE Control Systems Letters*, vol. 1, no. 2, pp. 340–345, 2017.

[18] K. Zhou, J. Doyle, and K. Glover, *Robust and Optimal Control*. Englewood Cliffs, NJ: Prentice-Hall, 1996.

[19] G. E. Dullerud and F. G. Paganini, *A Course in Robust Control Theory: a convex approach*. NY: Springer-Verlag, 1999.

[20] P. J. Antsaklis and A. N. Michel, *Linear Systems*. New York, NY: Birkhauser Boston, 2006.

[21] J. R. Silvester, "Determinants of block matrices," *The Mathematical Gazette*, vol. 84, no. 501, pp. 460–467, 2000.

[22] F. Yang, M. Gani, and D. Henrion, "Fixed-order robust H_∞ controller design with regional pole assignment," *IEEE Transactions on Automatic Control*, vol. 52, no. 10, pp. 1959–1963, 2007.

[23] N. S. Nise, *Control System Engineering*. New Delhi, India: John Willey and Sons, sixth ed., 2011.