Feedback Controller Norm Optimization for Linear Time Invariant Descriptor Systems with Pole Region Constraint

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***Abstract*—An algorithm is proposed to compute a state feedback gain matrix for a linear time invariant, regular descriptor system, which ensures that i) the closed loop system is impulse-free and ii) all the finite poles are placed within a *pre-defined* stability region in the complex plane. The associated design flexibilities are exploited in minimizing upper bounds of the Frobenius norm of respective gain matrices. A class of linear matrix inequality (LMI) regions in the complex plane are chosen as constraints for the closed loop finite poles. By representing a subset of the nonsingular matrices through solution of an LMI, and linearizing the set of matrix inequalities, arise in the regional pole (finite poles) assignment, the associated optimizations are formulated as semidefinite programs. The effectiveness of the developed algorithm is demonstrated through numerical examples. Significant reduction in the norm of feedback gain matrix is achieved in a two generator six bus power system.**

***Index Terms*—Descriptor systems, impulse elimination, finite pole placement, LMIs, convex optimizations.**

I. Introduction

Many physical systems, such as micro-grids, space vehicles and constrained mechanical systems are often modeled appropriately by combination of differential and algebraic equations. The differential equations describe the dynamic behavior of the energy storage elements while the algebraic equations represent the state constraints, boundary conditions and conservation laws [1], [2]. One of the key features that distinguishes such systems, referred to as *descriptor systems*, from the standard state space systems is that the time response might contain impulsive terms and its derivatives [3]. Significant effort has already been made to understand the associated theoretical underpinnings and its elimination from the response (see [4], [1], [3], [2], [5] and the references therein).

One of the effective ways to eliminate impulses from a linear time invariant (LTI) descriptor system is by designing a suitable state feedback gain matrix that reduces the *nilpotent index* to one (the poles at infinity are moved to some finite locations in the complex plane) [4], [3], [5] or *strangeness index* to zero [2]. From the applications perspective, in addition to the impulse elimination, it is also essential that the response of descriptor system should satisfy the specified transient bounds, such as settling time and damping ratio, to ensure safe operations. A conventional approach to achieve the latter objective is by assigning (with a feedback control) all the finite poles of the closed loop descriptor system at some specific locations in the complex plane [4], [3], [5].

The choice of feedback gain matrices for a descriptor system, that i) eliminate impulses from the response and ii) place the finite poles at some fixed locations in the complex plane, is not unique [3], [2], [5]. On the other hand, a limited magnitude of control signal can be provided to the actuators and the cost of the actuators grows quickly with increasing the magnitude of control signal. Since the control signal magnitude is directly proportional to the norm of the feedback gain matrix, the minimum feedback gain satisfying the desired objectives would be preferable. It has been observed in practical applications that the *fixed pole placement* approach produces high magnitude control signal. To address this, recently in [6], [7], *regional pole placement* paradigm is proposed. Such pole placement scenario is more relevant in practice since the performance specifications are usually mentioned in terms of time domain characteristics, such as settling time and damping ratio. Hence, it would be enough if a designed controller guarantees that all the closed loop poles are placed within some desirable region in the complex plane. This design philosophy offers extra flexibility on reducing the norm of feedback gain matrix.

Considering the above observations, in this article we propose convex algorithms to compute feedback gain matrices that are required for impulse elimination and assigning the finite poles within a *pre-defined* stability region in the complex plane. Addition of the resulting gain matrices will produce another gain matrix, which will ensure achieving combination of above objectives in the closed loop. We first obtain an impulse free descriptor system by formulating a semidefinite program that minimizes (upper bound) the Frobenius norm of gain matrix over a subset of nonsingular matrices. Then, following to a result proposed in [8] we compute a projection representation of the resulting impulse free system, which enables determining the solution of a descriptor system by adding the solution of a purely differential and a purely algebraic system. We show that the problem of assigning the finite poles is equivalent to place the eigenvalues of an appropriate submatrix associated with the purely differential equation in the projected system. Using a result [9] on LMI regions in the complex plane, and performing some appropriate linearization of the matrix inequalities, we formulate a semidefinite program to minimize an upper bound of the Frobenius norm of an associated feedback gain matrix. To the best of our knowledge, there is no result in the literature that combines the above

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objectives and formulated as convex optimizations.

Typical control objectives, such as regularization, impulse elimination and finite pole assignment for an LTI descriptor system is well studied in the literature (see [4], [3], [5], [2], [10], [11], [12] and the references therein). One of the traditional approaches to design a state feedback control for a descriptor system is by decoupling the original system into the fast (algebraic) and slow (differential) subsystems. To accomplish this, the system coefficient matrices are first transformed to *Weierstrass canonical form* [3], [5]. A similar decomposition (slow and fast subsystems) is achieved in [4] by defining the two special invariant subspaces. The design flexibilities associated with the feedback gain matrices are exploited in achieving robust pole assignment [13], [14], minimum norm gain matrix [12] and combination of both [15]. A non-convex algorithm, that uses the Sylvester equation, is proposed in [15] to assign all the closed loop poles (including infinite) at some fixed locations in the complex plane. Necessary and sufficient conditions are proposed in [16], [17] which ensure that the descriptor system is regular, impulse free and all the finite poles are confined within an LMI region in the complex plane. Additionally, a state feedback gain matrix is designed in [18], [19] which can place all the closed loop finite poles within a stability region. On the other hand, the algorithm proposed in this article emphasizes on minimizing the norm of the gain matrix (to reduce the magnitude of control signal significantly) with the constraints that the closed loop descriptor system is impulse-free and all the finite poles are placed within a *pre-defined* LMI region in the complex plane. The associated non-convex problems are formulated as convex optimizations (sub-optimal), which are computationally tractable. Recently, in [20], an impulse-free descriptor system is obtained by formulating a semi-definite program to compute the associated gain matrix. It is highlighted that the objective of only minimizing the norm of gain matrix may introduce numerical errors in the finite pole assignment. To overcome this, an alternative objective function is chosen where norm of the gain matrix and condition number of the associated nonsingular matrix are minimized simultaneously.

The rest of the paper is organized as follows. In Section II we formulate the problem following to some preliminaries on descriptor systems. In Section III we demonstrate a procedure to compute a gain matrix that transforms the original descriptor system into an impulse-free system. Following a projection approach, in Section IV we develop a convex algorithm to assign all the closed loop finite poles within a pre-defined stability region in the complex plane. Numerical examples are presented in Section V.

Notation: We denote the field of real numbers as $\mathbb{R}$, and the set of real matrices having m rows and n columns as $\mathbb{R}^{m\times n}$. The notation I is used to denote an identity matrix. For a matrix $A\in\mathbb{R}^{m\times n}$, $\text{rank}(A)$ and $\text{tr}(A)$ represent the rank of A and trace of A, respectively. The column-stacking operator of A is denoted as $\text{vec}(A)$ [21]. The condition number [22] of $A\in\mathbb{R}^{n\times n}$ is denoted as $\kappa(A)$, while $\det(A)$ represents the determinant of A. The largest and smallest singular values of A are represented as $s_1(A)$ and $s_n(A)$, respectively. For symmetric A, $\lambda_{max}(A)$ and $\lambda_{min}(A)$ denote the maximum and minimum eigenvalues. The notation $A\succ 0$ ($A\succeq 0$, $A\prec 0$) denotes that A is a symmetric positive definite (positive semi-definite, negative definite) matrix. The Frobenius norm of $A\in\mathbb{R}^{m\times n}$ is denoted as $\|A\|_F$, and defined as $\|A\|_F := \sqrt{\sum_{i=1}^m\sum_{j=1}^n |a_{ij}|^2}$.

II. Preliminaries and Problem Formulation

Let us consider an LTI continuous time descriptor system, represented by the following equation:

$$E\dot{x} = Ax + Bu, \tag{1}$$

where $E\in\mathbb{R}^{n\times n}$ is singular, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $x:\mathbb{R}\to\mathbb{R}^n$ is the state vector and $u:\mathbb{R}\to\mathbb{R}^m$ is an input to the system. We assume that system (1) is regular, that is, there exists a complex number $s\in\mathbb{C}$ such that $\det(sE-A)\neq 0$. We refer to the roots of polynomial $\det(sE-A)$ as *finite poles* of (1). We say system (1) is *impulse-free* if it has $d:=\text{rank}(E)$ finite poles (see the Appendix for relationship between the impulse-free and *semi-explicit index one* descriptor systems). We assume that $\text{rank}(E)=d$ and system (1) is not impulse-free, that is, (1) does not have d finite poles. Furthermore, we assume that (1) is C-controllable (see the Appendix).

Considering a linear state feedback control $u=Fx$ where $F\in\mathbb{R}^{m\times n}$, the closed loop system, associated with system (1), is given by

$$E\dot{x} = (A+BF)x. \tag{2}$$

Then, the objective of this work is to compute the feedback gain matrix F such that the closed loop system (2) has following properties: i) it is impulse-free and ii) all the finite poles, denoted as $\{\lambda_1,\lambda_2,\cdots,\lambda_d\}$, are placed within a stability region $\mathbb{S}$ in the complex plane. It is well known [3], [5] that if λ_i for $i=1,2,\cdots,d$ takes any complex value (including its conjugate) from the stability region $\mathbb{S}$, depicted in Figure 1, the response of (2) will satisfy the following transient bounds: i) settling time $t_s\leq\frac{4}{\delta}$ seconds and ii) damping ratio $\zeta\geq\cos\theta$ where $\delta>0$ and $\theta\in[0,\pi/2)$. We define $\mathbb{S}$, shown in Figure 1, as follows [9]:

$$\mathbb{S} := \{s\in\mathbb{C}\mid L(s)\prec 0\} \tag{3}$$

where

$$L(s) := \begin{bmatrix} s+\bar{s}+2\delta & 0 & 0\\ 0 & \sin\theta(s+\bar{s}) & \cos\theta(s-\bar{s})\\ 0 & \cos\theta(\bar{s}-s) & \sin\theta(s+\bar{s})\end{bmatrix}.$$

In order to compute the feedback gain matrix F, we proceed by first designing a state feedback control $u=\hat{F}x+\tilde{u}$ where $\hat{F}\in\mathbb{R}^{m\times n}$, which yields an impulse-free descriptor system

$$E\dot{x} = (A+B\hat{F})x + B\tilde{u}. \tag{4}$$

Note that (4) is a semi-explicit index one descriptor system. Then, another state feedback control $\tilde{u}=\tilde{F}x$ where $\tilde{F}\in\mathbb{R}^{m\times n}$ is designed to place all the finite poles of the closed loop system:

$$E\dot{x} = (A+B\hat{F}+B\tilde{F})x \tag{5}$$

within $\mathbb{S}$. The desired gain matrix F is then computed as follows: $F=\hat{F}+\tilde{F}$. Recall that in addition to the impulse

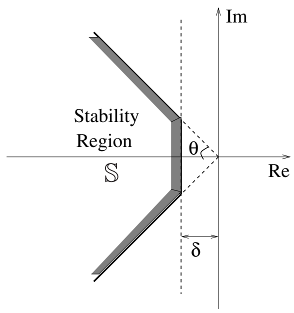

Figure 1. Stability region $\mathbb{S}$ in the complex plane. If all the finite poles of closed loop descriptor system (2) are clustered within $\mathbb{S}$ then its response will satisfy the bounds: i) settling time $t_s \leq \frac{4}{\delta}$ seconds and ii) damping ratio $\zeta \geq \cos\theta$.

elimination and finite pole assignment, we are also interested in minimizing $\|F\|_F$. Since $\|F\|_F = \|\hat{F}+\tilde{F}\|_F \leq \|\hat{F}\|_F + \|\tilde{F}\|_F$, we minimize $\|F\|_F$ by minimizing $\|\hat{F}\|_F$ and $\|\tilde{F}\|_F$ separately through the following two sub-problems.

Problem 1: Find $\hat{F}$ that minimizes $\|\hat{F}\|_F$ subject to the constraint that the closed loop system (4) is impulse-free.

Problem 2: Find $\tilde{F}$ that minimizes $\|\tilde{F}\|_F$ subject to the constraint that all the finite poles $\{\lambda_1, \lambda_2, \cdots, \lambda_d\}$ of (5) are placed within a *pre-defined* stability region $\mathbb{S}$, as defined in (3), in the complex plane.

Note that the gain matrix $\tilde{F}$ in Problem 2 is designed particularly to assign d finite poles of (5) within $\mathbb{S}$. Since $\mathrm{rank}(E) = d$, according to the definition, (5) is always impulse-free.

In the subsequent sections we formulate semidefinite programs, in particular, LMI optimizations to compute feedback gain matrices $\hat{F}$ and $\tilde{F}$.

III. Controller design for impulse elimination

Consider the descriptor system (1). Since E is singular, there exist (by performing a singular value decomposition (SVD) of E [22]) orthogonal matrices $Z \in \mathbb{R}^{n\times n}$ and $W \in \mathbb{R}^{n\times n}$ such that[1]

$$ZEW = \begin{bmatrix} \Sigma_e & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ ZAW = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ ZB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad (6)$$

where $\Sigma_e \in \mathbb{R}^{d\times d}$ is a diagonal matrix with positive diagonal entries, $A_{11} \in \mathbb{R}^{d\times d}$, $A_{12} \in \mathbb{R}^{d\times(n-d)}$, $A_{21} \in \mathbb{R}^{(n-d)\times d}$, $A_{22} \in \mathbb{R}^{(n-d)\times(n-d)}$, $B_1 \in \mathbb{R}^{d\times m}$ and $B_2 \in \mathbb{R}^{(n-d)\times m}$. By defining the feedback gain matrix $\hat{F}$ as

$$\hat{F} := \begin{bmatrix} \mathbf{0} & \hat{F}_2 \end{bmatrix} W^T, \quad (7)$$

where $\hat{F}_2 \in \mathbb{R}^{m\times(n-d)}$, we have

$$Z(A+B\hat{F})W = \begin{bmatrix} A_{11} & A_{12}+B_1\hat{F}_2 \\ A_{21} & A_{22}+B_2\hat{F}_2 \end{bmatrix}. \quad (8)$$

Considering the specific partitioning of the matrices in (6) and (8), it follows from [5, Theorem 7.3] that (4) is impulse-free if and only if $\det(A_{22}+B_2\hat{F}_2) \neq 0$. Hence, by defining a set $\mathscr{N}$ as follows:

$$\mathscr{N} := \left\{ \hat{F}_2 \in \mathbb{R}^{m\times(n-d)} \mid \det(A_F) \neq 0 \right\}$$

where $A_F := A_{22}+B_2\hat{F}_2$, the optimization associated with Problem 1 becomes:

Problem 3:

$$\min_{\hat{F}_2 \in \mathscr{N}} \ \|\hat{F}\|_F.$$

The solution of Problem 3 ensures that the closed loop system (4) is impulse-free. However, the associated nonsingular matrix A_F, which will be used in the finite pole assignment, might not be well-conditioned (see [20] and the discussion in Example 1). Furthermore, since the set $\mathscr{N}$ is non-convex, Problem 3 is a non-convex optimization. To overcome these difficulties, in the following result we compute a (convex) subset of $\mathscr{N}$ that can be represented as solutions of an LMI, a lower bound for the smallest singular value of A_F and upper bound for the largest singular value of A_F. These bounds will then be used in the optimization to improve the condition number of A_F.

Theorem 1: Let $\hat{F}$ be defined as in (7). If $\hat{F}_2$ satisfies

$$\text{(i)} \begin{bmatrix} \frac{1}{2}\left(A_F + A_F^T\right) & \hat{\beta}I \\ \hat{\beta}I & I \end{bmatrix} \succ 0,\ \text{(ii)} \begin{bmatrix} I & \mathrm{vec}(\hat{F}_2) \\ \mathrm{vec}(\hat{F}_2)^T & \hat{\gamma} \end{bmatrix} \succ 0 \quad (9)$$

for some non zero scalars $\hat{\beta}$ and $\hat{\gamma}$, then

a) A_F is nonsingular,
b) the Frobenius norm of $\hat{F}$ will satisfy $\|\hat{F}\|_F < \sqrt{\hat{\gamma}}$,
c) the bounds on $s_1(A_F)$ and $s_n(A_F)$ are:

$$s_1(A_F) \leq s_1(A_{22}) + s_1(B_2)\sqrt{\hat{\gamma}} \ \text{ and } \ s_n(A_F) \geq \hat{\beta}^2. \quad (10)$$

Proof: According to the Schur complement relation, the inequality (i) in (9) yields:

$$\frac{1}{2}\left(A_F + A_F^T\right) - \hat{\beta}^2 I \succ 0,$$

which implies $\lambda_{min}\left(\frac{1}{2}\left(A_F + A_F^T\right)\right) > \hat{\beta}^2$. Furthermore, it follows from [23, Corollary 3.1.5] that

$$s_n(A_F) \geq \lambda_{min}\left(\frac{1}{2}\left(A_F + A_F^T\right)\right).$$

Hence, $s_n(A_F) \geq \hat{\beta}^2$. Since $\hat{\beta}$ is a non zero scalar, $\hat{\beta}^2$ is positive, and hence $s_n(A_F)$ is positive. This shows that the matrix A_F is nonsingular.

Applying the Schur complement formula to the inequality (ii) in (9), we have

$$\hat{\gamma} - \mathrm{vec}(\hat{F}_2)^T \mathrm{vec}(\hat{F}_2) > 0.$$

This implies $\|\hat{F}_2\|_F^2 < \hat{\gamma}$ (since $\|\hat{F}_2\|_F^2 = \mathrm{vec}(\hat{F}_2)^T \mathrm{vec}(\hat{F}_2)$). Using (7) and the fact that W is orthogonal, we have

$$\begin{aligned} \|\hat{F}\|_F^2 &= \mathrm{tr}\left(\hat{F}^T\hat{F}\right) = \mathrm{tr}\left(\hat{F}\hat{F}^T\right) \\ &= \mathrm{tr}\left(\begin{bmatrix} 0 & \hat{F}_2 \end{bmatrix} W^T W \begin{bmatrix} 0 & \hat{F}_2 \end{bmatrix}^T\right) \\ &= \mathrm{tr}\left(\hat{F}_2\hat{F}_2^T\right) = \|\hat{F}_2\|_F^2 < \hat{\gamma}. \end{aligned}$$

Hence, $\|\hat{F}\|_F < \sqrt{\hat{\gamma}}$.

[1] Remark: In the singular value decomposition of a real matrix the orthogonal matrices are chosen as real matrices.

Recall that $A_F = A_{22} + B_2\hat{F}_2$. Hence, it follows from [23, Theorem 3.3.16] that

$$s_1(A_F) = s_1(A_{22} + B_2\hat{F}_2) \le s_1(A_{22}) + (s_1(B_2)s_1(\hat{F}_2)). \quad (11)$$

Note that $s_1(\hat{F}_2) = \|\hat{F}_2\|_2 \le \|\hat{F}_2\|_F$ [22]. We have already shown $\|\hat{F}_2\|_F < \sqrt{\hat{\gamma}}$, and hence $s_1(\hat{F}_2) \le \sqrt{\hat{\gamma}}$. Then, (11) will become $s_1(A_F) \le s_1(A_{22}) + s_1(B_2)\sqrt{\hat{\gamma}}$. The bound on $s_n(A_F)$ is already established. This completes the proof. ■

The inequalities in (9) are LMIs with parameters $\hat{F}_2$, $\hat{\gamma}$ and $\hat{\beta}$. Let us now define a set $\mathcal{N}_C$ as follows:

$$\mathcal{N}_C := \left\{\hat{F}_2 \in \mathbb{R}^{m\times(n-d)} \mid \text{LMI (i) in (9) holds for } \hat{\beta} \in \mathbb{R}\setminus 0\right\}.$$

Since $\mathcal{N}_C$ is the set of solutions of an LMI, it is a convex set [24]. Furthermore, $\mathcal{N}_C \subseteq \mathcal{N}$. Hence, to convexify Problem 3, we minimize $\|\hat{F}\|_F$ over a convex set $\mathcal{N}_C$. This modification will result a sub-optimal solution to Problem 3 (since $\mathcal{N}_C \subseteq \mathcal{N}$). The LMI (ii) in (9) ensures that $\|\hat{F}\|_F < \sqrt{\hat{\gamma}}$, and hence $\|\hat{F}\|_F$ can be minimized by minimizing $\hat{\gamma}$. Furthermore, to improve the condition number of A_F: $\kappa(A_F) = \frac{s_1(A_F)}{s_n(A_F)}$, that is, to make $\kappa(A_F)$ as small as possible, we need to maximize $s_n(A_F)$ and minimize $s_1(A_F)$, simultaneously. Now, from the relation (10) it can be observed that the minimization of $\hat{\gamma}$ will minimize $s_1(A_F)$, while maximization of $\hat{\beta}$ will maximize $s_n(A_F)$. This leads to the following optimization.

Problem 4:

$$\max_{\hat{F}_2,\hat{\beta},\hat{\gamma}} \quad \hat{\beta} - \hat{\gamma}$$

subject to

$$\text{(i)} \begin{bmatrix} \frac{1}{2}\left(A_F + A_F^T\right) & \hat{\beta}I \\ \hat{\beta}I & I \end{bmatrix} \succ 0, \ \text{(ii)} \begin{bmatrix} I & \text{vec}(\hat{F}_2) \\ \text{vec}(\hat{F}_2)^T & \hat{\gamma} \end{bmatrix} \succ 0,$$

with non zero scalars $\hat{\beta}$ and $\hat{\gamma}$.

The solution of Problem 4 ensures that (4) is impulse-free and A_F is well-conditioned. Furthermore, since the constraints (i) and (ii) in Problem 4 are LMIs and the objective function is linear, it is a semi-definite program, and hence can be solved with existing LMI solvers, such as *SeDuMi* [25]. Furthermore, the C-controllability (hence impulse controllable) assumption of (1) ensures that Problem 4 is always feasible. Once $\hat{F}_2$ is computed by solving Problem 4, the matrix $\hat{F}$ can be obtained from the relation (7).

IV. Controller design for regional pole placement

Let us denote the solution of Problem 4 as $\hat{F}_2^*$. Then, according to (7), $\hat{F}^* = \begin{bmatrix}\mathbf{0} & \hat{F}_2^*\end{bmatrix} W^T$, and hence, (4) can be rewritten as

$$E\dot{x} = (A + B\hat{F}^*)x + B\tilde{u}. \quad (12)$$

Pre-multiplication of the orthogonal matrix Z (used in (6)) in (12) yields

$$E_s\dot{x} = A_s x + B_s\tilde{u}, \quad (13)$$

where

$$E_s = \begin{bmatrix} E_1 \\ \mathbf{0} \end{bmatrix}, \ A_s = \begin{bmatrix} A_1 + B_1\hat{F}^* \\ A_2 + B_2\hat{F}^* \end{bmatrix}, \ B_s = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

Note that system (13) is an impulse-free descriptor system. Furthermore, the matrices $E_1 \in \mathbb{R}^{d\times n}$ and $A_2 + B_2\hat{F}^* \in \mathbb{R}^{(n-d)\times n}$ are of full row rank, that is, $\text{rank}(E_1) = d$ and $\text{rank}(A_2 + B_2\hat{F}^*) = n - d$. We will use this system to address Problem 2.

Considering a linear state feedback control $\tilde{u} = \tilde{F}x$, (13) can be written as

$$E_s\dot{x} = (A_s + B_s\tilde{F})x. \quad (14)$$

The objective of this section is to formulate an LMI optimization to compute $\tilde{F}$ that assigns all the d finite poles $\{\lambda_1, \lambda_2, \cdots, \lambda_d\}$ of (14) within a *pre-defined* stability region $\mathbb{S}$. To address this we use a recently developed result [8] to separate the state vector $x(t)$ into its differential and algebraic components by projecting it onto the appropriate subspaces. For completeness, we briefly discuss the result next.

Considering the impulse-free system (13), let us define the following two subspaces:

$$\mathbb{E}_d := \text{Im}(E_s^T), \ \mathbb{E}_a := \ker(E_s),$$

where Im and ker denote the *image* and *kernel*, respectively. Corresponding to these two subspaces, we can now partition the state x additively into two parts x_d and x_a, which we will call *differential and algebraic parts* of x, respectively. To obtain x_d and x_a, let us define the two projectors

$$P_d := E_s^+ E_s, \quad \text{and} \quad P_d' := I - E_s^+ E_s, \quad (15)$$

where E_s^+ is the Moore-Penrose inverse of E_s [22]. Note that P_d is an orthogonal projector onto the subspace $\mathbb{E}_d$, whereas P_d' is an orthogonal projector onto the subspace $\mathbb{E}_a$. Let us define

$$x_d := P_d x, \ x_a := P_d' x, \quad (16)$$

then we have $x = x_d + x_a$. In addition, let us define another projector Q onto $\text{Im}(E_s)$ via

$$Q := E_s E_s^+ \quad \text{and} \quad Q' = I - E_s E_s^+. \quad (17)$$

Then, the following result holds.

Theorem 2: [8] Let the projectors P_d and P_d' be defined as in (15) and the variables x_d and x_a as in (16). Then, $x = x_d + x_a$ is a solution of (13) if and only if x_d and x_a are solutions of the system

$$\dot{x}_d = G_d x_d + B_d\tilde{u}, \quad (18a)$$
$$x_a = G_a x_d + B_a\tilde{u}, \quad (18b)$$

where

$$G_a := -(Q'A_sP_d')^+(Q'A_sP_d), \ G_d := E_s^+A_s(P_d + G_a), \quad (19a)$$
$$B_a := -(Q'A_sP_d')^+B_s, \ B_d := E_s^+B_s + E_s^+A_sB_a. \quad (19b)$$

According to the above theorem the solution $x(t)$ of impulse-free system (13) can be obtained by solving the projected system (18) and performing $x(t) = x_d(t) + x_a(t)$. This tells that the desired transient performance of system (13) can be achieved by modifying the time response of projected system (18) with an appropriate feedback control. Before discussing on design of feedback control for the projected system (18), let us first establish the following result where we simplify the matrices given in (19).

Lemma 1: Consider the impulse-free system (13). Denote $\hat{A}_1 = A_1 + B_1\hat{F}^* \in \mathbb{R}^{d\times n}$, $\hat{A}_2 = A_2 + B_2\hat{F}^* \in \mathbb{R}^{(n-d)\times n}$. Then, there exist orthogonal matrices $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{n\times n}$, which are partitioned as

$$U = \overset{\quad d \qquad n-d}{\left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right]} \begin{array}{c} d \\ n-d \end{array}, \quad V = \overset{d \quad n-d}{\left[\begin{array}{c|c} V_1 & V_2 \end{array}\right]}\ n, \tag{20}$$

such that the matrices in (19) are as follows:

$$G_d = V\begin{bmatrix} G_{d_{11}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} V^T, \quad B_d = V\begin{bmatrix} B_{d_1} \\ \mathbf{0} \end{bmatrix}, \tag{21a}$$

$$G_a = -V\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ G_{a_{21}} & \mathbf{0} \end{bmatrix} V^T, \quad B_a = -V\begin{bmatrix} \mathbf{0} \\ B_{a_2} \end{bmatrix}, \tag{21b}$$

where

$$\begin{aligned} G_{d_{11}} &= \Sigma_1^{-1} U_{11}^T \left(\hat{A}_1 V_1 - \hat{A}_1 V_2 (\hat{A}_2 V_2)^{-1} \hat{A}_2 V_1\right) \in \mathbb{R}^{d\times d}, \\ B_{d_1} &= \Sigma_1^{-1} U_{11}^T, \left(B_1 - \hat{A}_1 V_2 (\hat{A}_2 V_2)^{-1} B_2\right) \in \mathbb{R}^{d\times m}, \\ G_{a_{21}} &= (\hat{A}_2 V_2)^{-1} \hat{A}_2 V_1 \in \mathbb{R}^{(n-d)\times d}, \\ B_{a_2} &= (\hat{A}_2 V_2)^{-1} B_2 \in \mathbb{R}^{(n-d)\times m}, \end{aligned}$$

and $\Sigma_1 \in \mathbb{R}^{d\times d}$ is a diagonal matrix with diagonal entries equal to the non-zero singular values of E_s.

Proof: See the Appendix. ■

With the feedback control $\tilde{u} = \tilde{F}x$, the projected system (18) will become

$$\dot{x}_d = G_d x_d + B_d \tilde{F} x, \quad x_a = G_a x_d + B_a \tilde{F} x. \tag{22}$$

Let us define the feedback gain matrix $\tilde{F}$ as

$$\tilde{F} := [\tilde{F}_1 \quad \mathbf{0}] V^T \tag{23}$$

where $\tilde{F}_1 \in \mathbb{R}^{m\times d}$. Then, with some calculations, it can be shown that $\tilde{F}x = \tilde{F}x_d$ (by using the definition of P_d and following the proof of Lemma 1). Hence,

$$\begin{aligned} \dot{x}_d &= (G_d + B_d\tilde{F})x_d, \\ &= V\begin{bmatrix} G_{d_{11}} + B_{d_1}\tilde{F}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} V^T x_d \end{aligned} \tag{24a}$$

$$\begin{aligned} x_a &= (G_a + B_a\tilde{F})x_d \\ &= -V\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ G_{a_{21}} + B_{a_2}\tilde{F}_1 & \mathbf{0} \end{bmatrix} V^T x_d. \end{aligned} \tag{24b}$$

In the following result we show that the finite poles of the closed loop system (14) are equal to the eigenvalues of the matrix $(G_{d_{11}} + B_{d_1}\tilde{F}_1)$, associated with the differential equation (24a).

Theorem 3: Consider the closed loop systems (14) and (24). Let us denote $\hat{G}_{d_k} := G_{d_{11}} + B_{d_1}\tilde{F}_1$. Then, the finite poles of system (14) are equal to the eigenvalues of $\hat{G}_{d_k}$, that is,

$$\Lambda\{\det(sE_s - A_s - B_s\tilde{F})\} = \Lambda\{\det(sI - \hat{G}_{d_k})\}$$

where $\Lambda\{\bullet\}$ denotes the roots of a polynomial.

Proof: Considering the notations used for matrices E_s, A_s, B_s and $\tilde{F}$, we have

$$\begin{aligned} &\Lambda\{\det\ (sE_s - A_s - B_s\tilde{F})\} \\ &= \Lambda\left\{\det\left(s\begin{bmatrix} E_1 \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\begin{bmatrix} \tilde{F}_1 & \mathbf{0} \end{bmatrix} V^T\right)\right\} \\ &= \Lambda\left\{\det\left(s\begin{bmatrix} E_1 \\ \mathbf{0} \end{bmatrix} V - \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} V - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\begin{bmatrix} \tilde{F}_1 & \mathbf{0} \end{bmatrix}\right)\right\} \\ &= \Lambda\left\{\det\left(s\begin{bmatrix} E_1 \\ \mathbf{0} \end{bmatrix} V - \begin{bmatrix} \hat{A}_1 V_1 & \hat{A}_1 V_2 \\ \hat{A}_2 V_1 & \hat{A}_2 V_2 \end{bmatrix} - \begin{bmatrix} B_1\tilde{F}_1 & \mathbf{0} \\ B_2\tilde{F}_1 & \mathbf{0} \end{bmatrix}\right)\right\}. \end{aligned}$$

Then, pre-multiplication of $\begin{bmatrix} I & -\hat{A}_1 V_2 (\hat{A}_2 V_2)^{-1} \\ \mathbf{0} & I \end{bmatrix}$ with the above matrices leads to

$$\begin{aligned} &\Lambda\{\det(sE_s - A_s - B_s\tilde{F})\} = \\ &\Lambda\left\{\det\left(s\begin{bmatrix} E_1 \\ \mathbf{0} \end{bmatrix} V - \begin{bmatrix} A_{V_{11}} & \mathbf{0} \\ \hat{A}_2 V_1 & \hat{A}_2 V_2 \end{bmatrix} - \begin{bmatrix} B_{V_1}\tilde{F}_1 & \mathbf{0} \\ B_2\tilde{F}_1 & \mathbf{0} \end{bmatrix}\right)\right\} \end{aligned} \tag{25}$$

where

$$A_{V_{11}} = \hat{A}_1 V_1 - \hat{A}_1 V_2 (\hat{A}_2 V_2)^{-1} \hat{A}_2 V_1, \tag{26a}$$

$$B_{V_1} = B_1 - \hat{A}_1 V_2 (\hat{A}_2 V_2)^{-1} B_2. \tag{26b}$$

Now, pre-multiply an orthogonal matrix U^T, to the matrices in (25), which satisfies $E_s = U\begin{bmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} V^T$. Then, we have

$$\begin{aligned} &\Lambda\{\det\ (sE_s - A_s - B_s\tilde{F})\} = \\ &\Lambda\left\{\det\left(s\begin{bmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - U^T\begin{bmatrix} A_{V_{11}} & \mathbf{0} \\ \hat{A}_2 V_1 & \hat{A}_2 V_2 \end{bmatrix} - U^T\begin{bmatrix} B_{V_1}\tilde{F}_1 & \mathbf{0} \\ B_2\tilde{F}_1 & \mathbf{0} \end{bmatrix}\right)\right\}. \end{aligned}$$

Considering the structure of $E_s = \begin{bmatrix} E_1 \\ \mathbf{0} \end{bmatrix}$, where E_1 is of full row rank, it follows from Proposition 1 (see the Appendix) that

$$U^T = \begin{bmatrix} U_{11}^T & \mathbf{0} \\ \mathbf{0} & U_{22}^T \end{bmatrix}.$$

The special structure of U^T leads to the following relations:

$$\begin{aligned} &\Lambda\{\det\ (sE_s - A_s - B_s\tilde{F})\} = \\ &\Lambda\left\{\det\left(s\begin{bmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} U_{11}^T A_{V_{11}} & \mathbf{0} \\ U_{22}^T \hat{A}_2 V_1 & U_{22}^T \hat{A}_2 V_2 \end{bmatrix} - \begin{bmatrix} U_{11}^T B_{V_1}\tilde{F}_1 & \mathbf{0} \\ U_{22}^T B_2\tilde{F}_1 & \mathbf{0} \end{bmatrix}\right)\right\}. \end{aligned}$$

Furthermore, pre-multiplication of a nonsingular matrix $\begin{bmatrix} \Sigma_1^{-1} & \mathbf{0} \\ \mathbf{0} & U_{22} \end{bmatrix}$ to the above matrices yields

$$\begin{aligned} &\Lambda\{\det(sE_s - A_s - B_s\tilde{F})\} = \\ &\Lambda\left\{\det\left(\begin{bmatrix} sI - \Sigma_1^{-1} U_{11}^T A_{V_{11}} - \Sigma_1^{-1} U_{11}^T B_{V_1}\tilde{F}_1 & \mathbf{0} \\ -\hat{A}_2 V_1 - B_2\tilde{F}_1 & -\hat{A}_2 V_2 \end{bmatrix}\right)\right\}. \end{aligned}$$

Now, using (26) and following Lemma 1, we have

$$\Lambda\{\det(sE_s - A_s - B_s\tilde{F})\} = \Lambda\left\{\det\left(\begin{bmatrix} sI - \hat{G}_{d_k} & \mathbf{0} \\ -\hat{A}_2 V_1 - B_2\tilde{F}_1 & -\hat{A}_2 V_2 \end{bmatrix}\right)\right\}.$$

Since $\hat{A}_2 V_2$ is nonsingular, the above relation leads to $\Lambda\{\det(sE_s - A_s - B_s\tilde{F})\} = \Lambda\{\det(sI - \hat{G}_{d_k})\}$. This completes the proof. ■

According to the above theorem the eigenvalues of $\hat{G}_{d_k}$ are equal to the finite poles of (14). Moreover,

$$\begin{aligned} \Lambda\left\{\det(sE_s - A_s - B_s\tilde{F})\right\} &= \Lambda\left\{\det(Z^T)\det(sE - A - B\hat{F}^* - B\tilde{F})\right\} \\ &= \Lambda\left\{\det(sE - A - BF)\right\}, \end{aligned}$$

where $F = \hat{F}^* + \tilde{F}$ and Z is an orthogonal matrix (see beginning of this section), and hence the finite poles of the closed loop systems (2) and (14) are also equal. This concludes that in order to place the finite poles of system (2) within a stability region $\mathbb{S}$, we need to design an appropriate feedback gain matrix $\tilde{F}_1$ which can assign all the eigenvalues of $\hat{G}_{d_k} = G_{d_{11}} + B_{d_1}\tilde{F}_1$ within $\mathbb{S}$. However, it follows from the linear system theory [26] that to place the eigenvalues of $\hat{G}_{d_k}$ at any arbitrary locations in the complex plane (with an appropriate $\tilde{F}_1$), following condition must hold: $\operatorname{rank}\left(\begin{bmatrix} B_{d_1} & G_{d_{11}}B_{d_1} & \cdots & G_{d_{11}}^{d-1}B_{d_1}\end{bmatrix}\right) = d$, equivalently, the pair $(G_{d_{11}}, B_{d_1})$ should be controllable. In the following result we show that the C-controllability assumption for system (1) will guarantee about it.

Theorem 4: Consider the descriptor system (1) and the projected system (18). Then, the following statements are equivalent.

1) Descriptor system (1) is C-controllable.
2) Following rank conditions hold:

$$\operatorname{rank}\left(\begin{bmatrix} B_{d_1} & G_{d_{11}}B_{d_1} & \cdots & G_{d_{11}}^{d-1}B_{d_1}\end{bmatrix}\right) = d \tag{27a}$$

$$\operatorname{rank}\left(\begin{bmatrix} G_{a_{21}}B_{d_1} & \cdots & G_{a_{21}}G_{d_{11}}^{n-d-1}B_{d_1} & B_{a_2}\end{bmatrix}\right) = n-d. \tag{27b}$$

Proof: See the Appendix. ■

Hence, according to Theorem 4 we can place the eigenvalues of $\hat{G}_{d_k}$ within a stability region $\mathbb{S}$ in the complex plane by designing an appropriate gain matrix $\tilde{F}_1$. Note that the stability region $\mathbb{S}$, shown in Figure 1, is an LMI region [9]. It is shown in [9] that all the eigenvalues $\hat{G}_{d_k}$ will be confined within $\mathbb{S}$ if and only if $\hat{G}_{d_k}$ satisfies the following matrix inequalities:

$$\hat{G}_{d_k}X + X\hat{G}_{d_k}^T + 2\delta X \prec 0,$$
$$\begin{bmatrix} \sin\theta(\hat{G}_{d_k}X + X\hat{G}_{d_k}^T) & \cos\theta(\hat{G}_{d_k}X - X\hat{G}_{d_k}^T) \\ \cos\theta(X\hat{G}_{d_k}^T - \hat{G}_{d_k}X) & \sin\theta(\hat{G}_{d_k}X + X\hat{G}_{d_k}^T)\end{bmatrix} \prec 0,$$

where $X \in \mathbb{R}^{d\times d}$ is a symmetric positive definite matrix. By replacing $\hat{G}_{d_k}$ with $G_{d_{11}} + B_{d_1}\tilde{F}_1$, the above matrix inequalities can be written as:

$$G_{d_{11}}X + B_{d_1}\tilde{F}_1X + XG_{d_{11}}^T + X\tilde{F}_1^TB_{d_1}^T + 2\delta X \prec 0, \tag{28a}$$
$$\begin{bmatrix} \sin\theta L_{11}(X,\tilde{F}_1) & \cos\theta(L_{21}(X,\tilde{F}_1))^T \\ \cos\theta L_{21}(X,\tilde{F}_1) & \sin\theta L_{11}(X,\tilde{F}_1)\end{bmatrix} \prec 0, \tag{28b}$$

where

$$L_{11}(X,\tilde{F}_1) = G_{d_{11}}X + B_{d_1}\tilde{F}_1X + XG_{d_{11}}^T + X\tilde{F}_1^TB_{d_1}^T,$$
$$L_{21}(X,\tilde{F}_1) = XG_{d_{11}}^T + X\tilde{F}_1^TB_{d_1}^T - G_{d_{11}}X - B_{d_1}\tilde{F}_1X.$$

It is clear from the above that if for some $\tilde{F}_1$ and $X \succ 0$ the matrix inequalities (28) hold then all the eigenvalues of $\hat{G}_{d_k}$, that is, the finite poles of (14) will be confined within $\mathbb{S}$. Hence, (28) can be used as a constraint while formulating an optimization for Problem 2. However, in (28), $\tilde{F}_1$ and X are free parameters, and hence the matrix inequalities are nonlinear in $\tilde{F}_1$ and X. To represent them as LMIs we introduce a new matrix variable $Y_1 = \tilde{F}_1X \in \mathbb{R}^{m\times d}$ and rewrite (28) in terms of free parameters X and Y_1 as follows:

$$G_{d_{11}}X + B_{d_1}Y_1 + XG_{d_{11}}^T + Y_1^TB_{d_1}^T + 2\delta X \prec 0, \tag{29a}$$
$$\begin{bmatrix} \sin\theta L_{11}(X,Y_1) & \cos\theta(L_{21}(X,Y_1))^T \\ \cos\theta L_{21}(X,Y_1) & \sin\theta L_{11}(X,Y_1)\end{bmatrix} \prec 0, \tag{29b}$$

where

$$L_{11}(X,Y_1) = G_{d_{11}}X + B_{d_1}Y_1 + XG_{d_{11}}^T + Y_1^TB_{d_1}^T,$$
$$L_{21}(X,Y_1) = XG_{d_{11}}^T + Y_1^TB_{d_1}^T - G_{d_{11}}X - B_{d_1}Y_1.$$

Now the matrix inequalities in (29) are linear with free parameters Y_1 and X, however, the gain matrix $\tilde{F}_1$ does not appear directly, and hence the direct optimization of $\|\tilde{F}_1\|_F = \|\tilde{F}\|_F$ (since V is orthogonal) will not be possible. Next, we establish a relationship between Y_1 and $\tilde{F}_1$ which enables to formulate an LMI optimization to minimize an upper bound of $\|\tilde{F}_1\|_F$.

Since $Y_1 = \tilde{F}_1X$, with some calculations it can be shown that

$$\operatorname{vec}(\tilde{F}_1^T) = X_d^{-1}\operatorname{vec}(Y_1^T) \tag{30}$$

where

$$X_d = \begin{bmatrix} X & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & X & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & X\end{bmatrix}.$$

Hence,

$$\begin{aligned}\|\tilde{F}_1\|_F^2 &= \operatorname{vec}(\tilde{F}_1^T)^T\operatorname{vec}(\tilde{F}_1^T) \\ &= \operatorname{vec}(Y_1^T)^T[X_dX_d^T]^{-1}\operatorname{vec}(Y_1^T)\end{aligned} \tag{31}$$

which is a non-convex function in the optimization variables Y_1 and X. To overcome the above difficulty we compute an upper bound for $\|\tilde{F}_1\|_F^2$ and then formulate an optimization in terms of the variables appearing in the resulting function. It is clear that $\|\tilde{F}_1\|_F = \|\operatorname{vec}(\tilde{F}_1^T)\|_2$. Hence, minimization of $\|\tilde{F}_1\|_F$ is equivalent to minimizing $\|\operatorname{vec}(\tilde{F}_1^T)\|_2$. From (30) we have

$$\|\operatorname{vec}(\tilde{F}_1^T)\|_2 = \|X_d^{-1}\operatorname{vec}(Y_1^T)\|_2 \le \|X_d^{-1}\|_2\|\operatorname{vec}(Y_1^T)\|_2. \tag{32}$$

To obtain convexity, we relax the requirements by minimizing the upper bound of $\|\operatorname{vec}(\tilde{F}_1^T)\|_2$, i.e. $\|\operatorname{vec}(Y_1^T)\|_2$ and $\|X_d^{-1}\|_2$ separately. Note that

$$\|X_d^{-1}\|_2 = \frac{1}{s_n(X_d)}.$$

Hence, $\|X_d^{-1}\|_2$ can be minimized by maximizing $s_n(X_d)$. This leads to the following result.

Theorem 5: If for some matrix $Y_1 \in \mathbb{R}^{m\times d}$ and nonzero scalars β, γ, following conditions:

$$\text{(i)}\quad \begin{bmatrix} X_d & \beta I \\ \beta I & I\end{bmatrix} \succ 0, \quad \text{(ii)}\quad X \succ 0,$$
$$\text{(iii)}\quad \begin{bmatrix} I & \operatorname{vec}(Y_1^T) \\ \operatorname{vec}(Y_1^T)^T & \gamma\end{bmatrix} \succ 0,$$
$$\text{(iv)}\quad G_{d_{11}}X + B_{d_1}Y_1 + XG_{d_{11}}^T + Y_1^TB_{d_1}^T + 2\delta X \prec 0,$$
$$\text{(v)}\quad \begin{bmatrix} \sin\theta L_{11}(X,Y_1) & \cos\theta(L_{21}(X,Y_1))^T \\ \cos\theta L_{21}(X,Y_1) & \sin\theta L_{11}(X,Y_1)\end{bmatrix} \prec 0,$$

hold, then all the eigenvalues of $\hat{G}_{d_k}$ will be clustered within the stability region $\mathbb{S}$. Furthermore, the Frobenius norm of $\tilde{F}$ will satisfy

$$\|\tilde{F}\|_F \leq \frac{\sqrt{\gamma}}{\beta^2}.$$

Proof: According to the Schur complement relation and following to the discussion in the proof of Theorem 1, LMIs (i) and (iii) yield $s_n(X_d) \geq \beta^2$ and $\|\text{vec}(Y_1^T)\|_2 < \sqrt{\gamma}$, respectively. Furthermore, since $\|\tilde{F}_1\|_F = \|\text{vec}(\tilde{F}_1^T)\|_2$, according to (32) we have

$$\|\tilde{F}_1\|_F \leq \|X_d^{-1}\|_2 \|\text{vec}(Y_1^T)\|_2 \leq \frac{\sqrt{\gamma}}{\beta^2}.$$

Using (23) and considering the fact that V is orthogonal, we have $\|\tilde{F}\|_F \leq \frac{\sqrt{\gamma}}{\beta^2}$ (see also the proof of Theorem 1). By replacing Y_1 with $\tilde{F}_1 X$, LMIs (iv) and (v) will become (28), and hence it follows from [9, Theorem 2.2] that all the eigenvalues of $\hat{G}_{d_k}$ are clustered in $\mathbb{S}$. ∎

According to Theorem 5, since $\|\tilde{F}\|_F \leq \frac{\sqrt{\gamma}}{\beta^2}$, we can indirectly minimize $\|\tilde{F}\|_F$ by formulating the following optimization:

Problem 5:

$$\max_{X, Y_1, \beta, \gamma} \quad \beta - \gamma$$

subject to

$$\text{(i)} \quad \begin{bmatrix} X_d & \beta I \\ \beta I & I \end{bmatrix} \succ 0, \quad \text{(ii)} \quad X \succ 0,$$

$$\text{(iii)} \quad \begin{bmatrix} I & \text{vec}(Y_1^T) \\ \text{vec}(Y_1^T)^T & \gamma \end{bmatrix} \succ 0,$$

$$\text{(iv)} \quad G_{d_{11}} X + B_{d_1} Y_1 + X G_{d_{11}}^T + Y_1^T B_{d_1}^T + 2\delta X \prec 0,$$

$$\text{(v)} \quad \begin{bmatrix} \sin\theta L_{11}(X, Y_1) & \cos\theta (L_{21}(X, Y_1))^T \\ \cos\theta L_{21}(X, Y_1) & \sin\theta L_{11}(X, Y_1) \end{bmatrix} \prec 0.$$

Note that all the constraints in Problem 5 are linear with respect to the optimization variables (X, Y_1, β, γ). Hence, it is an LMI optimization problem and can be solved with standard LMI solvers, like *SeDuMi* [25].

After solving Problem 5, the feedback gain matrix $\tilde{F}_1$ can be computed by using the relation (30). Then, $\tilde{F}$ can be obtained from the relation (23). Let us denote it as $\tilde{F}^*$, then the closed loop system (14) will become

$$E\dot{x} = (A + B\hat{F}^* + B\tilde{F}^*)x = (A + BF^*)x, \tag{33}$$

where $F^* = \hat{F}^* + \tilde{F}^*$. Note that the resulting system (33) is impulse-free, since there are d finite poles. Moreover, since all the finite poles are clustered within the stability region $\mathbb{S}$, the time response of (33) will satisfy the desired transient bounds (t_s and ζ).

Remark 1: Note that the constraints (iv) and (v) in Problem 5 arise so as to ensure that the response of closed loop system satisfies both the transient bounds: settling time and damping ratio. However, if settling time is the only specification then Problem 5 has to be solved without the constraint (v). Similarly, for damping ratio requirement, Problem 5 has to be solved without the constraint (iv).

Remark 2: Note that (13) is a semi-explicit index one system, and hence the decomposition in (35) will take the following form (see the Appendix for details):

$$SE_sT = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad SA_sT = \begin{bmatrix} J_s & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}. \tag{34}$$

By defining $\tilde{F} := [\tilde{F}_1 \ \mathbf{0}]T$ and writing $SB_s = [B_1^T \ B_2^T]^T$, the closed loop system (14) yields the following two subsystems:

$$\dot{x}_1 = (J_s + B_1\tilde{F}_1)x_1 \text{ and } \mathbf{0} = B_2\tilde{F}_1 x_1 + x_2.$$

It is well known [3], [5] that the finite poles of (14) are equal to the eigenvalues of $(J_s + B_1\tilde{F}_1)$. Hence, one may consider the procedure proposed in this section to design $\tilde{F}_1$ to place the closed loop finite poles within $\mathbb{S}$. However, this approach has the following limitations. It has been observed [27], [28] that often ill-conditioned transformation matrices S and T are required to compute the decomposition (34), and hence slight perturbations in the plant data may lead to largely perturbed subsystems. This situation may introduce numerical errors in the finite pole assignment. Since S and T are not orthogonal, $\|\tilde{F}\|_F = \|[\tilde{F}_1 \ \mathbf{0}]T\|_F \neq \|\tilde{F}_1\|_F$, and hence the minimization of $\|\tilde{F}_1\|_F$ and $\|\tilde{F}\|_F$ are not equal. Moreover, the non-uniqueness associated with the choice of transformation matrices S and T to obtain (34) (see the algorithm in [3]) gives no idea how to choose *a priori* a suitable set of S and T that can produce minimum $\|\tilde{F}\|_F$. The algorithm proposed in this section, however, overcomes these limitations.

The procedure to obtain the feedback gain matrix F^* is summarized in the following design steps.

Design Steps:

1) Verify if the given descriptor system is impulse-free. If it is so, then set $\hat{F}^* = \mathbf{0}$ and go to Step 4, else continue with Step 2.
2) Compute the matrices Z and W by performing a SVD of E. Then, compute ZEW, ZAW and ZB as in (6).
3) Solve Problem 4 and compute $\hat{F}^*$ using the relation (7).
4) Compute $G_{d_{11}}$ and $B_{d_{11}}$ according to Lemma 1.
5) Solve Problem 5 (see Remark 1) and compute $\tilde{F}_1{}^*$ and $\tilde{F}^*$ using the relations (30) and (23), respectively.
6) Compute $F^* = \hat{F}^* + \tilde{F}^*$.

In the following section we consider numerical examples to demonstrate the effectiveness of the developed algorithm.

V. Numerical Example

Example 1: Consider a system $E\dot{x} = Ax + Bu$ with following numerical data [5, Example 7.3]:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \ A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix};$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T.$$

A state feedback gain matrix F need to be designed to achieve the following objectives in the response of closed loop system:

i) it is impulse-free and ii) satisfies the transient bound: $t_s \leq 10$ seconds. The finite poles of the above system are: $-0.5 \pm 0.8660i$ and 1. It can be observed that the open loop system is not impulse-free (number of finite poles is strictly less than rank of E). To make it impulse-free we followed the design steps and obtained

$$\hat{F}_2^* = \begin{bmatrix} 0.0627 & 0.2601 \\ 0.0756 & 0.3135 \end{bmatrix},$$

by solving Problem 4. Then, using (7), we computed

$$\hat{F}^* = \begin{bmatrix} 0 & 0 & 0 & -0.2601 & 0 & 0.0627 \\ 0 & 0 & 0 & -0.3135 & 0 & 0.0756 \end{bmatrix},$$

with $\|\hat{F}^*\|_F = 0.4190$.

According to the procedure discussed in Section IV, the matrices associated with the projected system (18), are as follows:

$$G_{d_{11}} = \begin{bmatrix} 0 & 1 & 0.4534 & 0 \\ -0.2411 & 0.2411 & -1.6340 & 1 \\ 0.2411 & -0.2411 & 1.6340 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{d_1} = \begin{bmatrix} -0.5465 & 0.4534 \\ -1.8752 & -1.6340 \\ 1.8752 & 1.6340 \\ 0 & 0 \end{bmatrix}.$$

Since $t_s \leq 10$, all the eigenvalues of $G_{d_{11}} + B_{d_1}\tilde{F}_1$ should be placed left to the vertical line at -0.4 ($\delta = 0.4$) in the complex plane. For this, Problem 5 is solved without the constraint (v) (since ζ is not a performance bound) and obtained the following feedback gain matrix:

$$\tilde{F}_1^* = \begin{bmatrix} 1.1949 & 1.2649 & 0.6105 & 1.2166 \\ -1.3712 & -1.1961 & -1.8771 & -1.1901 \end{bmatrix}.$$

Corresponding to this gain matrix, the eigenvalues of $G_{d_{11}} + B_{d_1}\tilde{F}_1^*$ are: $-0.4001 \pm 0.2102i$ and $-0.4696 \pm 0.8605i$. Using (23), the gain matrix

$$\tilde{F}^* = \begin{bmatrix} -1.1949 & -1.2166 & -1.2649 & 0 & 0.6105 & 0 \\ 1.3712 & 1.1901 & 1.1961 & 0 & -1.8771 & 0 \end{bmatrix},$$

with $\|\tilde{F}^*\|_F = 3.6237$. We observed that the eigenvalues of $G_{d_{11}} + B_{d_1}\tilde{F}_1^*$ are equal to the finite poles of closed loop system $E\dot{x} = (A + BF^*)x$ where $F^* = \hat{F}^* + \tilde{F}^*$. Since there are four finite poles, the closed loop system is impulse-free. Furthermore, the location of finite poles confirms that the closed loop response will satisfy $t_s \leq 10$ seconds.

For completeness, we compare the condition number of the matrix A_F, obtained by solving Problem 4 and the following problem:

$$\min_{\hat{F}_2, \hat{\beta}, \hat{\gamma}} \quad \hat{\gamma}$$

subject to the constraints in (9). Note that the above optimization is formulated only to minimize $\|\hat{F}\|_F$. By solving the above optimization we obtained following gain matrix:

$$\hat{F}^* = 10^{-6} \times \begin{bmatrix} 0 & 0 & 0 & -0.2535 & 0 & -0.0059 \\ 0 & 0 & 0 & -0.2573 & 0 & 0.0054 \end{bmatrix},$$

where we set $\hat{\beta} \geq 0.0001$. With this gain matrix, we observed that the matrix A_F is nonsingular, and hence the associated closed loop system is impulse-free. Furthermore, we computed the Frobenius norm of $\hat{F}^*$ and condition number of A_F as $\|\hat{F}^*\|_F = 3.6127 \times 10^{-7}$ and $\kappa(A_F) = 1.3844 \times 10^6$, respectively. Although with this setting we could reduce the norm of gain matrix significantly, the condition number of A_F has increased to a large value, in comparison to the results: $\|\hat{F}^*\|_F = 0.4190$ and $\kappa(A_F) = 1.1703$ obtained by solving Problem 4.

Example 2: In this example we consider a linearized model of 6-bus, 2-generator power system, as depicted in Figure 2. The numerical data for E, A and B, associated with this system, can be found in [29] (the first topology is considered). It is a 10^{th} order model, and the open loop finite poles are at: 0, $-0.6997 \pm 0.5118i$ and -1.6006. Since the number of finite poles is equal to the rank of E ($d = 4$), it is an impulse-free system. The objective is to design a feedback control to achieve the following transient bounds on the response of closed loop system: i) settling time $t_s \leq 20$ seconds and ii) damping ratio $\zeta \geq 0.7$. Corresponding to these data the parameters for the stability region $\mathbb{S}$ would be $\delta = 0.2$ and $\theta = 45^o$. To compute the gain matrix we followed the procedure discussed in Section IV and obtained

$$\tilde{F}_1^* = \begin{bmatrix} -0.0031 & -0.0017 & -0.0031 & -0.0009 \\ -0.0031 & -0.0017 & -0.0031 & -0.0009 \\ -0.0031 & -0.0017 & -0.0031 & -0.0009 \\ -0.0031 & -0.0017 & -0.0031 & -0.0009 \\ -0.0031 & -0.0017 & -0.0031 & -0.0009 \\ -0.0031 & -0.0017 & -0.0031 & -0.0009 \\ -0.0031 & -0.0017 & -0.0031 & -0.0009 \\ -0.0031 & -0.0017 & -0.0031 & -0.0009 \end{bmatrix},$$

with $\|\tilde{F}_1^*\|_F = 0.0136$, by solving Problem 5. Hence, $\tilde{F}^* = [\tilde{F}_1^* \;\; \mathbf{0}]V^T$ and $\|\tilde{F}^*\|_F = 0.0136$. Since the system is impulse-free, $\hat{F}^* = \mathbf{0}$, and hence $F^* = \tilde{F}^*$. The associated closed loop finite poles are -0.20002, $-0.6997 \pm 0.5118i$ and -1.6007. Since all the closed loop finite poles are inside the stability region $\mathbb{S}$, we achieved the specified transient performance.

We compare our results with a conventional method discussed in [3], [5], where all the closed loop finite poles need to be specified to achieve desired transient performance. The considered power system model is impulse-free, and hence, we only need to design a feedback gain matrix which can assign the closed loop finite poles at specified locations in the complex plane. Following the procedure discussed in [5, Chapter 8] we compute a feedback gain matrix to place the closed loop finite poles at -0.20002, $-0.6997 \pm 0.5118i$ and -1.6007, which are resulted in the proposed algorithm. It is observed that the Frobenius norm of the resulting gain matrix,

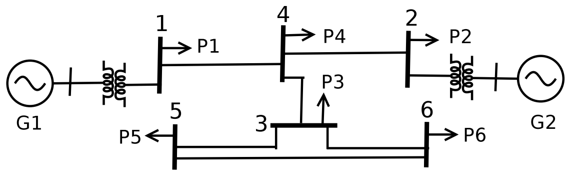

Figure 2. A 6-bus, 2-generator power system. In this topology, 1 and 2 are generator buses, 3, 4, 5 and 6 are load buses.

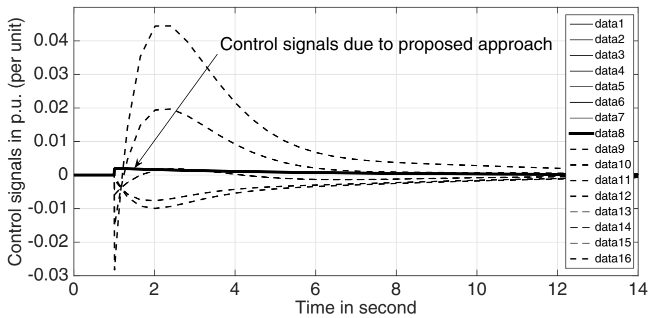

Figure 3. Comparison between the maximum overshoot of the control signals. The simulated control signals $u_i(t)$ for $i = 1, 2, \cdots, 8$ with proposed and conventional approaches are represented in solid lines and dashed lines, respectively. Note that there are sixteen control signals (eight for proposed approach and another eight for conventional approach), however, due to the overlapping, all are not clearly visible.

denoted as F_c, is $\|F_c\|_F = 0.1109$ where $F_c = \begin{bmatrix} F_{c_1} & \mathbf{0} \end{bmatrix}$ with

$$F_{c_1} = \begin{bmatrix} -0.0215 & -0.0626 & -0.0665 & 0.0256 \\ -0.0021 & 0.0142 & 0.0145 & -0.0018 \\ -0.0125 & -0.0273 & -0.0292 & 0.0130 \\ -0.0029 & 0.0109 & 0.0111 & -0.0006 \\ -0.0061 & -0.0018 & -0.0024 & 0.0039 \\ -0.0061 & -0.0018 & -0.0024 & 0.0039 \\ -0.0061 & -0.0018 & -0.0024 & 0.0039 \\ -0.0061 & -0.0018 & -0.0024 & 0.0039 \end{bmatrix}.$$

Comparing it with $\|\tilde{F}^*\|_F = 0.0136$, it can be observed that around 87.73% reduction in the norm of gain matrix is achieved by proposed algorithm.

Furthermore, to compare the magnitude of control signals ($|u_i(t)|$ for $i = 1, 2, \cdots, 8$), simulation of the considered power system is performed in MATLAB *Simulink*, by first transforming the descriptor model into an ordinary state space model (see the procedure discussed in [30]). Then, the transformed fourth order model is driven by the feedback gain matrices (modified F^* and F_c) obtained by proposed approach and conventional approach. The control signals are depicted in Figure 3. It can be observed that the magnitude of control signal in the proposed approach has reduced significantly in comparison to conventional approach. Furthermore, all the four state trajectories, depicted in Figure 4, have been converged to zero within the specified time limit, that is, 20 seconds.

VI. Conclusion

In this work we develop a novel algorithm to compute a state feedback gain matrix for an LTI regular descriptor system.

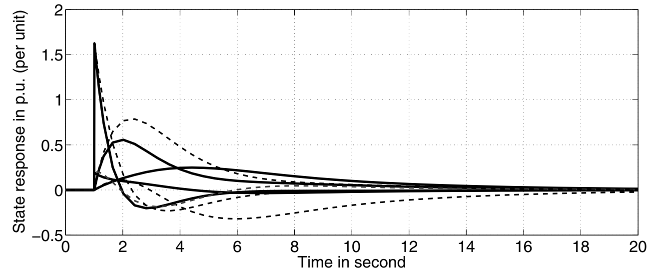

Figure 4. State response of the closed loop power system. The state trajectories due to proposed and conventional approaches are represented in solid lines and dashed lines, respectively. The simulation is done in MATLAB Simulink. The closed loop system is excited with a step input of step length 0.01 seconds.

The resulting sub-optimal gain matrix guarantees that all the finite poles of the closed loop system are placed within a *pre-defined* stability region in the complex plane, while at the same time, ensures that the response of the closed loop system is impulse-free. The advantages of the proposed algorithm are: i) one has to solve convex, in particular, LMI optimizations to compute the gain matrices and ii) it allows to place the closed loop finite poles within a *pre-defined* stability region in the complex plane, and hence significant reduction in the magnitude of feedback gain matrix occurs, which is evident from a power system example.

The conservatism of the proposed algorithm lies in the relaxation of nonlinear constraints and the non convex set, and hence produces a sub-optimal solution. Additionally, since the algorithm does not take account of robustness of the closed loop finite poles, they might be sensitive to the perturbation in the system parameters. While these objectives are under current investigation, the proposed algorithm seems working nicely with the practical examples.

Appendix

A. Preliminaries on descriptor systems:

Corresponding to the pair (E, A) in (1), there exist two nonsingular matrices $S \in \mathbb{R}^{n \times n}$ and $T \in \mathbb{R}^{n \times n}$ such that

$$SET = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & N \end{bmatrix}, \quad SAT = \begin{bmatrix} J & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \tag{35}$$

where $J \in \mathbb{R}^{d \times d}$ and $N \in \mathbb{R}^{(n-d)\times(n-d)}$ is a nilpotent matrix [3], [1]. The index ν of N ($N^\nu = 0$ and $N^{\nu-1} \neq 0$) in the decomposition (35) (*Weierstrass canonical form*) is called (differentiation) *index* of the descriptor system (1) [1]. If $N = \mathbf{0}$ in (35), then the descriptor system is called *semi-explicit index one* system [1]. One of the key aspects of a semi-explicit index one system is that no solutions of it contain impulses (assuming $u(t)$ does not have derivatives of Dirac impulses), and hence is called *impulse-free* descriptor system [3], [5].

Note that (35) yields following two subsystems:

$$\dot{x}_1 = Jx_1 + B_1 u, \quad N\dot{x}_2 = x_2 + B_2 u \tag{36}$$

where $T^{-1}x = [x_1^T \ x_2^T]^T$ and $SB = [B_1^T \ B_2^T]^T$. We then have the following definitions.

Definition 1: [5, Chapter 5] A descriptor system (1) is *completely controllable* (C-controllable) if, for any $t_1 > 0$, $[x_1^T(t_0)\ x_2^T(t_0)]^T \in \mathbb{R}^n$ and a vector $k \in \mathbb{R}^n$, there exists an admissible (sufficiently smooth) control input $u(t)$ such that $[x_1^T(t_1)\ x_2^T(t_1)]^T = k$.

Definition 2: [5, Chapter 5] A vector $k \in \mathbb{R}^n$ is said to be *reachable* if there exist an initial condition $x_1(t_0)$, an admissible control input $u(t)$ and a finite time $t_1 > 0$ such that $[x_1^T(t_1)\ x_2^T(t_1)]^T = k$.

By denoting the set of reachable points as $\mathscr{R}$, a descriptor system is said to be *$\mathscr{R}$-controllable* if it is completely controllable in the reachable set $\mathscr{R}$ [5]. Similarly, the *impulse controllability* of a descriptor system characterizes the ability to eliminate impulses from the response via an admissible control input $u(t)$ (refer to [3], [5] for explicit definitions). A descriptor system is C-controllable implies that it is $\mathscr{R}$-controllable and impulse controllable [5].

B. Proposition and Proof of Theorems:

Proposition 1: Let us consider the matrix $E_s = \begin{bmatrix} E_1^T & \mathbf{0} \end{bmatrix}^T$ where E_1 is full row rank. Assume that E_s satisfies the following decomposition (by performing SVD of E_s):

$$E_s = U \begin{bmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} V^T \tag{37}$$

where U and V are orthogonal matrices, and $\Sigma_1 \in \mathbb{R}^{d\times d}$ is a diagonal matrix with diagonal entries equal to the non-zero singular values of E_s. Then, the matrix U has the following structure

$$U = \begin{bmatrix} U_{11} & \mathbf{0} \\ \mathbf{0} & U_{22} \end{bmatrix}. \tag{38}$$

Proof: From (37) we have the following relation:

$$\begin{bmatrix} E_{11} & E_{12} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} U_{11}\Sigma_1 V_{11}^T & U_{11}\Sigma_1 V_{21}^T \\ U_{21}\Sigma_1 V_{11}^T & U_{21}\Sigma_1 V_{21}^T \end{bmatrix},$$

and hence $U_{21}\Sigma_1 \begin{bmatrix} V_{11}^T & V_{21}^T \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \end{bmatrix}$. Post-multiplication of $\begin{bmatrix} V_{11}^T & V_{21}^T \end{bmatrix}^T$ on both sides leads to $U_{21}\Sigma_1 = \mathbf{0}$. Since Σ_1 is nonsingular, we have $U_{21} = \mathbf{0}$. Furthermore, since $UU^T = I$, we have:

$$U_{11}U_{21} + U_{12}U_{22}^T = \mathbf{0} \text{ and } U_{21}U_{21}^T + U_{22}U_{22}^T = I.$$

Since, $U_{21} = \mathbf{0}$, the above relations become $U_{12}U_{22}^T = \mathbf{0}$ and $U_{22}U_{22}^T = I$. Then, by post multiplying U_{22} on both sides of $U_{12}U_{22}^T = \mathbf{0}$, we have $U_{12} = \mathbf{0}$. ■

Proof: Proof of Lemma 1: Since $E_s = \begin{bmatrix} E_1^T & \mathbf{0} \end{bmatrix}^T \in \mathbb{R}^{n\times n}$ and $\text{rank}(E_s) = d$, we can find orthogonal matrices U and V such that the decomposition in (37) holds. Hence,

$$E_s^+ = V \begin{bmatrix} \Sigma_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} U^T.$$

Then, the projector matrices, defined in (15), are given by

$$P_d = E_s^+ E_s = V \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} V^T \text{ and } P_d' = I - E_s^+ E_s = V \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} V^T.$$

Similarly, the projector matrix Q in (17) will take the following form:

$$Q = E_s E_s^+ = U \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} U^T.$$

Using (38) and identifying the fact that $U_{11}U_{11}^T = I$, Q and Q' will become:

$$Q = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \text{ and } Q' = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}.$$

This leads to the following relation

$$\begin{aligned} Q' A_s P_d' &= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} V^T \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{A}_2 V_2 \end{bmatrix} V^T. \end{aligned}$$

Since $\hat{A}_2$ is full row rank (recall that system (13) is impulse-free, and hence $\hat{A}_2$ is full row rank) and V_2 is full column rank, $\hat{A}_2 V_2$ is nonsingular. Hence, $(Q' A_s P_d')^+ = V \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\hat{A}_2 V_2)^{-1} \end{bmatrix}$. Furthermore,

$$\begin{aligned} Q' A_s P_d &= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} V^T \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \hat{A}_2 V_1 & \mathbf{0} \end{bmatrix} V^T. \end{aligned}$$

Then, the matrices G_a and B_a are as follows:

$$\begin{aligned} G_a &= -(Q' A_s P_d')^+ (Q' A_s P_d) \\ &= -V \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ (\hat{A}_2 V_2)^{-1} \hat{A}_2 V_1 & \mathbf{0} \end{bmatrix} V^T \quad \text{and} \end{aligned} \tag{39}$$

$$\begin{aligned} B_a &= -(Q' A_s P_d')^+ B_s \\ &= -V \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\hat{A}_2 V_2)^{-1} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = -V \begin{bmatrix} \mathbf{0} \\ (\hat{A}_2 V_2)^{-1} B_2 \end{bmatrix}. \end{aligned} \tag{40}$$

Using (39) and considering the appropriate partitions of U and V as in (20), we compute the matrix G_d as follows:

$$\begin{aligned} G_d &= E_s^+ A_s (P_d + G_a) \\ &= V \begin{bmatrix} \Sigma_1^{-1} U_{11}^T & \Sigma_1^{-1} U_{12}^T \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{A}_1 V_1 - \hat{A}_1 V_2 (\hat{A}_2 V_2)^{-1} \hat{A}_2 V_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} V^T \\ &= V \begin{bmatrix} G_{d_{11}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} V^T, \end{aligned}$$

where $G_{d_{11}} = \Sigma_1^{-1} U_{11}^T \left(\hat{A}_1 V_1 - \hat{A}_1 V_2 (\hat{A}_2 V_2)^{-1} \hat{A}_2 V_1 \right) \in \mathbb{R}^{d\times d}$. Similarly we compute the matrix B_d as follows:

$$\begin{aligned} B_d &= E_s^+ B_s + E_s^+ A_s B_a = E_s^+ (B_s + A_s B_a) \\ &= V \begin{bmatrix} \Sigma_1^{-1} U_{11}^T & \Sigma_1^{-1} U_{12}^T \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} B_1 - \hat{A}_1 V_2 (\hat{A}_2 V_2)^{-1} B_2 \\ 0 \end{bmatrix} = V \begin{bmatrix} B_{d_1} \\ \mathbf{0} \end{bmatrix}, \end{aligned}$$

where $B_{d_1} = \Sigma_1^{-1} U_{11}^T \left(B_1 - \hat{A}_1 V_2 (\hat{A}_2 V_2)^{-1} B_2 \right) \in \mathbb{R}^{d\times m}$. This completes the proof. ■

Proof: Proof of Theorem 4: Assume that system (1) is C-controllable. We will show that the rank conditions in (27) hold. Since (1) is C-controllable, the set of all state reachable points from the origin is $\mathscr{R}_0 = \mathbb{R}^n$ [3], [5]. It is shown in [31] that the set $\mathscr{R}_0$ can also be computed from the the projected system (18). Before proceeding further, let us first define the following matrices:

$$\mathscr{C}_d := \begin{bmatrix} B_d & G_d B_d & \cdots & G_d^{n-1} B_d \end{bmatrix},$$
$$\mathscr{C}_{d_1} := \begin{bmatrix} B_{d_1} & G_{d_{11}} B_{d_1} & \cdots & G_{d_{11}}^{n-1} B_{d_1} \end{bmatrix},$$
$$\mathscr{C}_a := \begin{bmatrix} G_a B_d & G_a G_d B_d & \cdots & G_a G_d^{n-1} B_d & B_a \end{bmatrix},$$
$$\mathscr{C}_{a_2} := \begin{bmatrix} G_{a_{21}} B_{d_1} & G_{a_{21}} G_{d_{11}} B_{d_1} & \cdots & G_{a_{21}} G_{d_{11}}^{n-1} B_{d_1} & B_{a_2} \end{bmatrix}.$$

Then, by defining the following two subspaces:

$$\mathscr{X}_d := \text{Im } \mathscr{C}_d, \text{ and } \mathscr{X}_a := \text{Im } \mathscr{C}_a,$$

it follows from [31] that the reachable space $\mathscr{R}_0$ is

$$\mathscr{R}_0 = \mathscr{X}_d + \mathscr{X}_a.$$

Moreover, following Lemma 1, we can write

$$\text{rank}(\mathscr{C}_d) = \text{rank}\left(V \begin{bmatrix} \mathscr{C}_{d_1} \\ \mathbf{0} \end{bmatrix} \right) = \text{rank}\left(\begin{bmatrix} \mathscr{C}_{d_1} \\ \mathbf{0} \end{bmatrix} \right).$$

Since the zero block row in $\begin{bmatrix} \mathscr{C}_{d_1}^T & \mathbf{0} \end{bmatrix}^T$ does not contribute to the rank, we have

$$\text{rank}\left(\begin{bmatrix} \mathscr{C}_{d_1} \\ \mathbf{0} \end{bmatrix} \right) = \text{rank}(\mathscr{C}_{d_1}).$$

Recall that $B_{d_1} \in \mathbb{R}^{d \times m}$ and $G_{d_{11}} \in \mathbb{R}^{d \times d}$, hence $\text{rank}(\mathscr{C}_{d_1}) \leq d$. Moreover, according to the Cayley-Hamilton theorem [32]

$$\text{rank}\left(\mathscr{C}_{d_1}\right) = \text{rank}\left(\begin{bmatrix} B_{d_1} & G_{d_{11}} B_{d_1} & \cdots & G_{d_{11}}^{d-1} B_{d_1} \end{bmatrix} \right),$$

and hence

$$\text{rank}\left(\begin{bmatrix} B_{d_1} & G_{d_{11}} B_{d_1} & \cdots & G_{d_{11}}^{d-1} B_{d_1} \end{bmatrix} \right) \leq d.$$

Similarly it can be shown that $\text{rank}(\mathscr{C}_a) = \text{rank}(\mathscr{C}_{a_2})$, and hence according to the Cayley-Hamilton theorem

$$\text{rank}\left([G_{a_{21}} B_{d_1} \;\; G_{a_{21}} G_{d_{11}} B_{d_1} \;\; \cdots \;\; G_{a_{21}} G_{d_{11}}^{n-d-1} B_{d_1} \;\; B_{a_2}] \right) \leq n-d.$$

Since $\mathscr{R}_0 = \mathbb{R}^n$, we have $\dim(\mathscr{X}_d + \mathscr{X}_a) = n$, which implies $\dim \mathscr{X}_d + \dim \mathscr{X}_a - \dim(\mathscr{X}_d \cap \mathscr{X}_a) = n$ (dim refers to the dimension of a vector space). Hence, to ensure $\mathscr{R}_0 = \mathbb{R}^n$, the rank conditions in (27) must hold.

Now assume that the rank conditions in (27) hold. Then, it is clear that $\dim \mathscr{X}_d = d$ and $\dim \mathscr{X}_a = n-d$. We will show that $\dim(\mathscr{X}_d \cap \mathscr{X}_a) = 0$. Let us consider an arbitrary vector $\hat{x}$ such that $\hat{x} \in (\mathscr{X}_d \cap \mathscr{X}_a)$. Since $\hat{x} \in \mathscr{X}_d$, there exists a vector $\eta_d \in \mathbb{R}^{mn}$ such that $\mathscr{C}_d \eta_d = \hat{x}$ which implies

$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \mathscr{C}_{d_1} \\ \mathbf{0} \end{bmatrix} \eta_d = V_1 \mathscr{C}_{d_1} \eta_d = \hat{x}. \tag{41}$$

Similarly, since $\hat{x} \in \mathscr{X}_a$, there exists a vector $\eta_a \in \mathbb{R}^{m(n+1)}$ such that $\mathscr{C}_a \eta_a = \hat{x}$. This implies

$$V_2 \mathscr{C}_{a_2} \eta_a = \hat{x}. \tag{42}$$

From (41) and (42), we have $V_1 \mathscr{C}_{d_1} \eta_d = V_2 \mathscr{C}_{a_2} \eta_a$. Then, pre-multiplying V_1^T in both sides leads to $\mathscr{C}_{d_1} \eta_d = 0$. Then, it follows from (41) that $\hat{x} = 0$. Since $\hat{x}$ is arbitrary, we have $\mathscr{X}_d \cap \mathscr{X}_a = 0$, and hence $\dim(\mathscr{X}_d \cap \mathscr{X}_a) = 0$. This completes the proof. ■

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