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Consensus in Multi-agent Systems: A Transfer Function Based Controller Design Approach

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***Abstract*—In this article, the problem of output consensus in a multi-agent system is considered, where a distributed dynamic output feedback control architecture is proposed. The control architecture consists of two components, one is local controllers for the agents, and another is a gain for the network. The transfer function of local controller is designed based on the number of positive real axis (in the complex plane) zeros of the agent transfer function, and then network gain is selected by sketching the root-locus of a unity feedback system. It is shown that the output consensus can be achieved with an *integral* type controller for a class of agents, irrespective of their order. Moreover, with the proposed control architecture, there is no need of communication between the local controllers. The effectiveness of the developed approach is demonstrated through numerical examples.**

***Index Terms*—Linear systems, multi-agent systems, transfer function, output feedback.**

I. Introduction

In the recent decades, there has been a growing interest among the researchers to develop methodologies for coordinated control of a group of systems, referred to as *multi-agent system*, instead of a single unit. This is due to the demand in various application fields, such as smart power-grid, defense and space sectors [1], [2], where the deployment of a group of systems is preferred, which can work in a coordinated manner to achieve some global objectives. A multi-agent system (MAS) consists of a group of dynamical systems, referred to as *agents*, and a communication network for information exchange between the agents. A preferred control architecture for a MAS is *distributed feedback control*, where the local controllers for the agents are distributed over the entire network, and they produce appropriate control signals for the agents by taking relative information (state and/or output) from the neighborhood agents. The major global control objectives that are considered in the literature (see [2]–[4] and the references therein) are: i) consensus, ii) leader-follower synchronization and iii) formation.

In this work, we propose a methodology to design a distributed output feedback control architecture for a MAS to achieve *output consensus*, that is, the outputs of all the agents in the network become equal at the steady state. For this, we assume that the agents are identical, and they are single-input single-output (SISO) dynamical systems with linear time-invariant state space dynamics. By obtaining the agent transfer function $P(s)$, the controller transfer function $K(s)$ is constructed based on the number of zeros of $P(s)$, which lie on the positive real axis of the complex plane. Then, a *network gain (γ_c)* is chosen by using the root-locus of a unity negative feedback system, and the largest eigenvalue of the network graph Laplacian matrix. The combined implementation of $K(s)$ and γ_c ensures achieving output consensus in the network. With this control architecture, the *state consensus* is also achieved in the network.

Significant effort has already been made to address the problem of consensus in a MAS, for instance, see [3]–[15] and the references therein. For the agents with general linear time-invariant state space dynamics, distributed observer based output feedback control is proposed to achieve state consensus [9], [11], [16], [17]. Such observer based feedback control needs communication between the neighborhood agents as well as neighborhood observers. To reduce the communication links, which helps in reducing the overall implementation cost, observer type output feedback control is proposed in [18]–[20], where there is no need of interaction between the observers/controllers. However, in such observer based control architecture, the order of individual local controller is equal to the order of the agent. Motivated by the work of [18]–[20], in this work, we propose a distributed dynamic output feedback control for the MAS, where there is no need of interaction between the local controllers. Additionally, the proposed control architecture has following advantages. Since most of the physical systems have a very small number of positive real axis zeros in the complex plane, the proposed controller order is smaller than the order of the agent. In fact, if there are no positive real axis zeros of $P(s)$, then the controller $K(s)$ is an *integrator*. Since the controller order is small, it is easy to implement in the applications. Further, the controller synthesis procedure is very simple, where only classical control design approach is used.

Notations: I_n: identity matrix of size $n \times n$, $\mathbf{0}$: a matrix with zero entries, $\otimes$: Kronecker product, $\mathbb{C}$: the complex plane, $\mathbb{C}^-$: open left half of $\mathbb{C}$ (excluding imaginary axis), $\mathbb{C}^+$: open right half of $\mathbb{C}$ (excluding imaginary axis), diag$\{\bullet\}$: a diagonal matrix and blkdiag$\{\bullet\}$: a block diagonal matrix.

II. Main Results

Consider a MAS, which consists of a group of r identical agents with i^{th} agent, denoted as P_i, dynamics is as follows:

$$\dot{x}_i = A_p x_i + b_p u_i, \quad y_i = c_p x_i, \quad \forall\ i \in \mathcal{N}, \tag{1}$$

where $\mathcal{N} := \{1, 2, \cdots, r\}$ is a set, $x_i \in \mathbb{R}^n$ is the state vector, $u_i \in \mathbb{R}$ is the (control) input to P_i and $y_i \in \mathbb{R}$ is the output of P_i. Assume that the initial state of (1) is $x_i(0) \in \mathbb{R}^n$, and (1) is both controllable and observable, that is, the pair (A_p, b_p) is controllable and the pair (A_p, c_p) is observable, for all $i \in \mathcal{N}$.

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We represent the considered MAS with an undirected (connected) graph $\mathscr{G}$ on the set of r vertices: $\{v_1, v_2, \cdots, v_r\}$, where i^{th} vertex v_i corresponds to the i^{th} agent P_i. Further, an undirected edge between the two vertices v_i and v_j with weight $a_{ij} > 0$ corresponds to the communication link between agents: P_i and P_j. The agent P_i is said to be *neighbor* of P_j, if there exists an edge between vertices v_i and v_j in $\mathscr{G}$. The neighborhood set of i^{th} agent is then defined as follows: $N(i) := \{j \neq i \mid \text{there is an edge between } v_i \text{ and } v_j \text{ in } \mathscr{G}\}$. Corresponding to the graph $\mathscr{G}$, we associate a symmetric matrix $\mathscr{A}$, whose $(i,j)^{th} = (j,i)^{th}$ element is a_{ij}, if there is an undirected edge between the vertices v_i and v_j. Assuming that $\mathscr{G}$ has no self-loops, that is, the weight $a_{ii} = 0$, the diagonal elements of $\mathscr{A}$ are zeros. The matrix $\mathscr{A}$ is referred to as *adjacency matrix* of $\mathscr{G}$. The degree of i^{th} vertex $\delta(i)$ and the *degree matrix* Δ of $\mathscr{G}$ are defined as: $\delta(i) := \sum_{j\in N(i)} a_{ij}$ and $\Delta := \text{diag}\{\delta(1), \delta(2), \cdots, \delta(r)\}$, respectively. Then, we associate a symmetric matrix: $L := \Delta - \mathscr{A}$ with $\mathscr{G}$, which is referred to as *Laplacian matrix* of $\mathscr{G}$.

Assume that each agent P_i in the MAS has its own local controller K_i, whose state space representation is as follows:

$$\dot{\bar{x}}_i = A_k \bar{x}_i + b_k \varepsilon_i, \quad u_i = c_k \bar{x}_i, \quad \forall\ i \in \mathscr{N}, \tag{2}$$

where $\bar{x}_i \in \mathbb{R}^m$ is the state vector, $\varepsilon_i \in \mathbb{R}$ is the input to K_i and $u_i \in \mathbb{R}$ is the output from K_i, with initial state $\bar{x}_i(0) \in \mathbb{R}^m$. Let (2) be both controllable and observable, for all $i \in \mathscr{N}$. For $t \geq 0$ and $i \in \mathscr{N}$, we define the signal $\varepsilon_i(t)$ in (2) as follows:

$$\varepsilon_i(t) := \gamma_c \left[\sum_{j\in N(i)} a_{ij}(y_j(t) - y_i(t)) \right], \tag{3}$$

where γ_c is a positive real number, which is referred to as *network gain*. Note that $\varepsilon_i(t)$ is defined by considering all the neighborhood relative output information available for i^{th} controller K_i. Define: $\varepsilon := [\varepsilon_1\ \varepsilon_2\ \cdots\ \varepsilon_r]^T$ and $y := [y_1\ y_2\ \cdots\ y_r]^T$. Then, (3) can compactly be written as: $\varepsilon = -\gamma_c L y$, where L is the Laplacian matrix of $\mathscr{G}$. Define $\bar{L}_g := \gamma_c L$, and denote $\bar{\lambda}_i$, for $i \in \mathscr{N}$, as an eigenvalue of $\bar{L}_g$. Since the network graph $\mathscr{G}$ is connected, the Laplacian matrix L has $r-1$ non-zero eigenvalues and one 0 eigenvalue [4]. Hence, $\bar{L}_g$ has also $r-1$ non-zero eigenvalues, which are denoted as: $\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_{r-1}$ and one 0 eigenvalue, which is denoted as $\bar{\lambda}_r$.

Notice that the combination of (1), (2) and (3) form a closed loop system, where our interest is to achieve consensus among the outputs $y_i(t)$ of the agents. For a given set of initial states $x_i(0) \in \mathbb{R}^n$ and $\bar{x}_i(0) \in \mathbb{R}^m$, we say that consensus among the outputs, referred to as *output consensus*, is achieved in the MAS at some finite point ρ (non-zero real number), if $y_1(t) = y_2(t) = \cdots = y_r(t) = \rho$ as $t \to \infty$. To achieve output consensus, we use transfer function based design approach to obtain controller K_i and network gain γ_c. Since the agents have identical dynamics (1), they have same transfer function: $P(s) = c_p(sI_n - A_p)^{-1}b_p$. Let $P(s)$ be represented as follows:

$$P(s) = \frac{\beta(s)}{\alpha(s)} = \frac{\beta_{n-1}s^{n-1} + \ldots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0}. \tag{4}$$

Let $K(s)$ be the transfer function of each local controller K_i, which is represented as follows:

$$K(s) = \frac{\psi(s)}{\phi(s)} = \frac{\psi_{m-1}s^{m-1} + \cdots + \psi_1 s + \psi_0}{s^m + \phi_{m-1}s^{m-1} + \cdots + \phi_1 s + \phi_0}. \tag{5}$$

We then have the following result.

Theorem 1: Assume that the transfer function $P(s)$, as in (4), is stable, that is, the roots of $\alpha(s)$ are in $\mathbb{C}^-$, and $\beta_0 \neq 0$. Further, assume that $P(s)$ has $m-1$ number of zeros on the real axis of $\mathbb{C}^+$. For $i = 1, 2, \cdots, m-1$, let z_{c_i} and p_{c_i} be some positive and negative real numbers, respectively, such that p_{c_i} does not coincide with the zeros of $P(s)$. Using p_{c_i} and z_{c_i}, construct $K(s)$ as follows:

$$K(s) = \frac{\psi(s)}{\phi(s)} = \frac{\psi(s)}{s\widetilde{\phi}(s)} = \frac{(s - z_{c_1})(s - z_{c_2})\cdots(s - z_{c_{m-1}})}{s(s - p_{c_1})(s - p_{c_2})\cdots(s - p_{c_{m-1}})}, \tag{6}$$

and draw the root-locus of unity feedback system, as shown in Figure 1. Let $\overline{\kappa}$ be a positive real number such that for all values of κ in the interval: $(0, \overline{\kappa}]$, the branches of root-locus belong to $\mathbb{C}^-$. Then, for $\gamma_c \leq \frac{\overline{\kappa}}{\lambda_{\max}}$, the following set of polynomials are stable (roots are in $\mathbb{C}^-$):

$$\sigma_l(s) := \alpha(s)\phi(s) + \bar{\lambda}_l \beta(s)\psi(s), \quad \text{for } l = 1, 2, \cdots, r-1, \tag{7}$$

where $\lambda_{\max}$ is the largest eigenvalue of Laplacian matrix L and $\bar{\lambda}_l$ is an eigenvalue of $\bar{L}_g$.

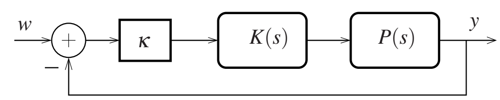

Figure 1: Unity (negative) feedback system

Proof: Corresponding to $P(s)$ and $K(s)$, define a transfer function: $F(s) = P(s)K(s) = \frac{\beta(s)\psi(s)}{\alpha(s)\phi(s)}$. Recall that system (1) is both controllable and observable, and hence, the polynomial pair $(\alpha(s), \beta(s))$ is co-prime (no common factors between $\alpha(s)$ and $\beta(s)$). Further, since the roots of $\psi(s)$ and $\widetilde{\phi}(s)$ are in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively, the pair $(\phi(s), \psi(s))$ is also co-prime. According to the assumption $\beta_0 \neq 0$, and hence, $P(s)$ does not have zeros at origin (0). Further, z_{c_i}'s and p_{c_i}'s are chosen such that there will be no common factors between the pairs: $(\alpha(s), \psi(s))$ and $(\beta(s), \widetilde{\phi}(s))$. Hence, $F(s)$ has no pole-zero cancellations. Moreover, since $P(s)$ and $K(s)$ are strictly proper, and degree of $\alpha(s)\phi(s)$ is $n+m$, the degree of characteristic polynomial $\bar{\sigma}(s) := \alpha(s)\phi(s) + \kappa\beta(s)\psi(s)$ of the unity feedback closed loop system, as in Figure 1, is $n+m$.

Note that the root-locus of unity feedback system in Figure 1 is the locus of roots of the polynomial $\bar{\sigma}(s)$ with respect to the gain $\kappa \geq 0$. The branches of root-locus starts ($\kappa = 0$) from the poles of $F(s)$ and ends ($\kappa \to \infty$) at the zeros (including zeros at ∞) of $F(s)$ [21], [22]. Since the degree of $\bar{\sigma}(s)$ is $n+m$, the root-locus has $n+m$ branches. Notice that $F(s)$ has all the poles in $\mathbb{C}^-$, except a pole at origin. Hence, except the branch that starts from origin, all the remaining $n+m-1$ branches of root-locus start from $\mathbb{C}^-$. The branch that starts from origin may either move along the positive or negative directions of real axis in $\mathbb{C}$. However, notice that $F(s)$ has even number of

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zeros $(2(m-1))$ on the real axis of $\mathbb{C}^+$. Hence, there will be no real axis segment of root-locus in between origin and the closest real axis zero of $F(s)$ in $\mathbb{C}^+$ (the angular contribution due to all poles and zeros within this real axis segment is zero degree [21]). This implies, the root-locus branch that starts from 0 will move towards $\mathbb{C}^-$. Similarly, when $P(s)$ has no real axis zero in $\mathbb{C}^+$, the controller $K(s)$ becomes $\frac{1}{s}$, and the branch starting from origin will move towards $\mathbb{C}^-$ to satisfy the angle criterion. Since the root-locus is continuous in nature (with respect to κ), it follows that there exists a $\overline{\kappa}$ such that for all real values of κ in the interval: $(0,\overline{\kappa}]$ the branches of root-locus belong to $\mathbb{C}^-$. Since γ_c is chosen such that $\gamma_c \leq \frac{\overline{\kappa}}{\lambda_{\max}}$, we have: $(0,\gamma_c\lambda_{\max}] \subseteq (0,\overline{\kappa}]$. Further, for $i=1,2,\cdots,r$, the eigenvalues $\bar{\lambda}_i$'s of $\bar{L}_g$ satisfy: $\bar{\lambda}_i=\gamma_c\lambda_i$, where λ_i's are the eigenvalues of L. Hence, for $l=1,2,\cdots,r-1$, we have: $\bar{\lambda}_l \in (0,\gamma_c\lambda_{\max}]$. Since the branches of root-locus belong to $\mathbb{C}^-$ for $\kappa \in (0,\overline{\kappa}]$, and the following set inclusion relation holds: $(0,\gamma_c\lambda_{\max}] \subseteq (0,\overline{\kappa}]$, by comparing with the polynomial $\bar{\sigma}(s)=\alpha(s)\phi(s)+\kappa\beta(s)\psi(s)$, it is clear that the roots of $\sigma_l(s)$, as defined in (7), are located in $\mathbb{C}^-$, and hence, they are all stable. ■

We will now provide a result where it is shown that the controller $K(s)$ and network gain γ_c, designed according to Theorem 1, ensure achieving output consensus (also state consensus) in the network. Since the agent dynamics (1) is controllable, one can obtain the following similar (equivalent) system representation of (1) using a non-singular transformation matrix P (see [23, Chapter 3] for the construction of P):

$$\dot{\xi}_i=\tilde{A}_p\xi_i+\tilde{b}_pu_i,\ y_i=\tilde{c}_p\xi_i \quad \forall\ i\in\mathcal{N}, \tag{8}$$

where $\tilde{A}_p=PA_pP^{-1}$, $\tilde{b}_p=Pb$ and $\tilde{c}_p=c_pP^{-1}$ with

$$\tilde{A}_p=\begin{bmatrix}-\alpha_{n-1} & \cdots & -\alpha_1 & -\alpha_0\\ 1 & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 1 & 0\end{bmatrix},\ \tilde{b}_p=\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix},\ \tilde{c}_p=\begin{bmatrix}\beta_{n-1}\\ \beta_{n-2}\\ \vdots\\ \beta_0\end{bmatrix}^T. \tag{9}$$

Similarly, assume that the controller $K(s)$, designed according to Theorem 1, has a state space realization (2). Since the polynomial pair $(\phi(s),\psi(s))$ is co-prime, and $K(s)$ is strictly-proper, a state space realization of the form (2), which is both controllable and observable, can be obtained for $K(s)$. Then, (2) can be transformed to the following similar system using a non-singular transformation matrix Q:

$$\dot{\bar{\xi}}_i=\tilde{A}_k\bar{\xi}_i+\tilde{b}_k\varepsilon_i,\quad u_i=\tilde{c}_k\bar{\xi}_i,\quad \forall\ i\in\mathcal{N}, \tag{10}$$

where $\tilde{A}_k=QA_kQ^{-1}$, $\tilde{b}_k=Qb_k$ and $\tilde{c}_k=c_kQ^{-1}$ with

$$\tilde{A}_k=\begin{bmatrix}-\phi_{m-1} & \cdots & -\phi_1 & 0\\ 1 & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 1 & 0\end{bmatrix},\ \tilde{b}_k=\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix},\ \tilde{c}_k=\begin{bmatrix}\psi_{m-1}\\ \psi_{m-2}\\ \vdots\\ \psi_0\end{bmatrix}^T. \tag{11}$$

Note that (8) and (10) have transfer functions $P(s)$ and $K(s)$ as in (4) and (5), respectively. We then have the following result.

Theorem 2: For a given agent dynamics (1) with its transfer function $P(s)$ as in (4), let $K(s)$ and γ_c be designed according to Theorem 1 such that the polynomials $\sigma_l(s)$, defined in (7), are stable. Let (2) be the state space realization of designed $K(s)$. Then, for $x_i(0)\in\mathbb{R}^n$ and $\bar{x}_i(0)\in\mathbb{R}^m$, the outputs of the agents satisfy the following: $y_1(t)=y_2(t)=\cdots=y_r(t)=\rho$, where ρ is a non-zero real number, and the states of the agents satisfy: $x_1(t)=x_2(t)=\cdots=x_r(t)=x^*\in\mathbb{R}^n$ as $t\to\infty$.

Proof: Corresponding to (1) and (2), define $\mathbf{x}\in\mathbb{R}^{rn}$ and $\bar{\mathbf{x}}\in\mathbb{R}^{rm}$ as: $\mathbf{x}:=[x_1^T\ x_2^T\ \cdots\ x_r^T]^T$ and $\bar{\mathbf{x}}:=[\bar{x}_1^T\ \bar{x}_2^T\ \cdots\ \bar{x}_r^T]^T$, respectively. By writing $x_{cl}=[\mathbf{x}^T\ \bar{\mathbf{x}}^T]^T$, the closed loop (combination of (1), (2) and (3)) state space equation can be written as [24]: $\dot{x}_{cl}=A_{cl}x_{cl}$, where $A_{cl}=\begin{bmatrix}I_r\otimes A_p & I_r\otimes b_pc_k\\ -\bar{L}_g\otimes b_kc_p & I_r\otimes A_k\end{bmatrix}$. Since the pairs (A_p,b_p) and (A_k,b_k) are controllable, and (A_p,c_p) and (A_k,c_k) are observable, it follows from [24, Theorem 1, Theorem 2] that the eigenvalues of A_{cl} are equal to the union of roots of the polynomials: i) $\sigma_l(s)$, as defined in (7), and ii) $\sigma_r(s)=s\alpha(s)\widetilde{\phi}(s)$. Note that the roots of $\sigma_l(s)$, $\alpha(s)$ and $\widetilde{\phi}(s)$ are in $\mathbb{C}^-$. Hence, A_{cl} has only one eigenvalue at origin (0), and others are in $\mathbb{C}^-$.

Since the pairs (A_p,b_p) and (A_k,b_k) are controllable, there exist non-singular transformation matrices P and Q such that the similar systems (8) and (10) of (1) and (2), respectively, can be obtained. Using P and Q, define the following non-singular matrix: $M:=\begin{bmatrix}I_r\otimes P & \mathbf{0}\\ \mathbf{0} & I_r\otimes Q\end{bmatrix}$. Then, corresponding to the closed loop system: $\dot{x}_{cl}=A_{cl}x_{cl}$, following similar system can be obtained: $\dot{\xi}_{cl}=\tilde{A}_{cl}\xi_{cl}$, where $\tilde{A}_{cl}=MA_{cl}M^{-1}=\begin{bmatrix}I_r\otimes\tilde{A}_p & I_r\otimes\tilde{b}_p\tilde{c}_k\\ -\bar{L}_g\otimes\tilde{b}_k\tilde{c}_p & I_r\otimes\tilde{A}_k\end{bmatrix}$, $(\tilde{A}_p,\tilde{b}_p,\tilde{c}_p)$ are as in (9), and $(\tilde{A}_k,\tilde{b}_k,\tilde{c}_k)$ are as in (11). Now, define the following vectors:

$$\mathbf{1}_r:=\begin{bmatrix}1 & \cdots & 1 & 1\end{bmatrix}^T\in\mathbb{R}^r,\ e_n:=\begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}^T\in\mathbb{R}^n,$$

$$e_m:=\begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}^T\in\mathbb{R}^m,\qquad \hat{h}_{cl}:=\begin{bmatrix}\frac{1}{\alpha_0}(\mathbf{1}_r\otimes e_n)\\ \frac{1}{\psi_0}(\mathbf{1}_r\otimes e_m)\end{bmatrix}.$$

Then, we have the following relation:

$$\begin{aligned}\tilde{A}_{cl}\hat{h}_{cl}&=\begin{bmatrix}I_r\otimes\tilde{A}_p & I_r\otimes\tilde{b}_p\tilde{c}_k\\ -\bar{L}_g\otimes\tilde{b}_k\tilde{c}_p & I_r\otimes\tilde{A}_k\end{bmatrix}\begin{bmatrix}\frac{1}{\alpha_0}(\mathbf{1}_r\otimes e_n)\\ \frac{1}{\psi_0}(\mathbf{1}_r\otimes e_m)\end{bmatrix}\\ &=\begin{bmatrix}\left(\mathbf{1}_r\otimes\frac{1}{\alpha_0}\tilde{A}_pe_n\right)+\left(\mathbf{1}_r\otimes\frac{1}{\psi_0}\tilde{b}_p\tilde{c}_ke_m\right)\\ \left(-\bar{L}_g\mathbf{1}_r\otimes\frac{1}{\alpha_0}\tilde{b}_k\tilde{c}_pe_n\right)+\left(\mathbf{1}_r\otimes\frac{1}{\psi_0}\tilde{A}_ke_m\right)\end{bmatrix}.\end{aligned} \tag{12}$$

Considering the matrices and vectors in (9) and (11), it is easy to notice that $\frac{1}{\alpha_0}\tilde{A}_pe_n=-e_1$ and $\frac{1}{\psi_0}\tilde{b}_p\tilde{c}_ke_m=e_1$, where $e_1=[1\ 0\ \cdots\ 0]^T\in\mathbb{R}^n$. Further, since $\bar{L}_g\mathbf{1}_r=0$ and $\frac{1}{\psi_0}\tilde{A}_ke_m=0$, it is then clear from (12) that $\tilde{A}_{cl}\hat{h}_{cl}=0$, and hence, $\hat{h}_{cl}$ belongs to the null space of $\tilde{A}_{cl}$. Since $\tilde{A}_{cl}=MA_{cl}M^{-1}$, the vector $\bar{h}_{cl}:=M^{-1}\hat{h}_{cl}$ belongs to the null space of A_{cl}. Hence, $h_{cl}:=\frac{1}{\|\bar{h}_{cl}\|_2}\bar{h}_{cl}$ ($\|.\|_2$: vector 2-norm), is a right eigenvector of A_{cl}, corresponding to its zero eigenvalue. Let g_{cl}^T be the left eigenvector of A_{cl} corresponding to its zero eigenvalue, such that $g_{cl}^Th_{cl}=1$. Since A_{cl} has only one eigenvalue at 0, and others are in $\mathbb{C}^-$, it can be shown (using modal decomposition, as in [4, Chapter 3]), that $x_{cl}(t)$ will converge to $x_{cl}^*=(g_{cl}^Tx_{cl}(0))h_{cl}$ as $t\to\infty$. Define $y:=[y_1\ y_2\ \cdots\ y_r]^T$. Then, the output $y(t)=\begin{bmatrix}I_r\otimes c_p & \mathbf{0}\end{bmatrix}x_{cl}(t)$ will converge to:

$$\begin{aligned}y^*&=\begin{bmatrix}I_r\otimes c_p & \mathbf{0}\end{bmatrix}x_{cl}^*=\frac{(g_{cl}^Tx_{cl}(0))}{\|\bar{h}_{cl}\|_2}\begin{bmatrix}I_r\otimes c_p & \mathbf{0}\end{bmatrix}\bar{h}_{cl}\\ &=\frac{(g_{cl}^Tx_{cl}(0))}{\|\bar{h}_{cl}\|_2}\begin{bmatrix}I_r\otimes c_p & \mathbf{0}\end{bmatrix}\begin{bmatrix}\frac{1}{\alpha_0}\left(\mathbf{1}_r\otimes P^{-1}e_n\right)\\ \frac{1}{\psi_0}\left(\mathbf{1}_r\otimes Q^{-1}e_m\right)\end{bmatrix}\\ &=\frac{(g_{cl}^Tx_{cl}(0))}{\|\bar{h}_{cl}\|_2\alpha_0}(\mathbf{1}_r\otimes\tilde{c}_pe_n)=\rho\mathbf{1}_r,\end{aligned} \tag{13}$$

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where $\rho = \frac{(g_{cl}^T x_{cl}(0))\beta_0}{\|\bar{h}_{cl}\|_2 \alpha_0}$. This implies, as $t \to \infty$, the outputs: $y_1(t) = y_2(t) = \cdots = y_r(t) = \rho$. Hence, the consensus among the outputs of agents is achieved. Furthermore, since $x_{cl}^* = (g_{cl}^T x_{cl}(0))h_{cl}$, we have the following relation:

$$x_{cl}^* = \frac{(g_{cl}^T x_{cl}(0))}{\|\bar{h}_{cl}\|_2}\bar{h}_{cl} = \frac{(g_{cl}^T x_{cl}(0))}{\|\bar{h}_{cl}\|_2}\begin{bmatrix} \frac{1}{\alpha_0}(\mathbf{1}_r \otimes P^{-1}e_n) \\ \frac{1}{\psi_0}(\mathbf{1}_r \otimes Q^{-1}e_m) \end{bmatrix}.$$

It is then clear that x_i will converge to $x^* = \frac{(g_{cl}^T x_{cl}(0))}{\|\bar{h}_{cl}\|_2 \alpha_0}P^{-1}e_n$ as $t \to \infty$, for all $i \in \mathcal{N}$. Hence, consensus among the states of the agents is also achieved. ■

Note that in Theorem 1, we give a synthesis procedure for computing the local controller $K(s)$ and network gain γ_c to stabilize a set of polynomials. We use these designed $K(s)$ and γ_c in Theorem 2 to prove that the consensus among the outputs (also states) of the agents will be achieved, when a state space realization of $K(s)$ and γ_c are implemented in the closed loop MAS. In the following section, we demonstrate the developed results with numerical examples.

III. Demonstrative examples

Example 1: Consider a group of four agents, forming a networked MAS, where the communication link between the agents is as per the graph $\mathcal{G}$, depicted in Figure 2-A. The dynamics of each agent is of the form (1), where $A_p = \left[\begin{smallmatrix} -2 & 1 \\ 3 & -4 \end{smallmatrix}\right]$, $b_p = [1 \quad 4]^T$, $c_p = [1 \quad 0]$. Corresponding to these data, the transfer function of the agent is: $P(s) = \frac{\beta(s)}{\alpha(s)} = \frac{s+8}{s^2+6s+5}$. Note that $P(s)$ satisfies the assumptions of Theorem 1. We now design $K(s)$ and γ_c to achieve output consensus. Since there are no zeros of $P(s)$ on the real axis of $\mathbb{C}^+$, it follows from Theorem 1 that $K(s) = \frac{1}{s}$. Then, the root-locus of the unity feedback system in Figure 1 is drawn, and is depicted in Figure 3. From the root-locus, $\overline{\kappa}$ is chosen as $\overline{\kappa} = 0.744$, according to Theorem 1. Since the largest eigenvalue of the Laplacian matrix L is 3.4142, we have: $\gamma_c = \frac{0.744}{3.4142} = 0.2179$. Then, the matrix $\bar{L}_g$ is computed, and its eigenvalues are: $\bar{\lambda}_1 = 0.7440$, $\bar{\lambda}_2 = 0.4358$, $\bar{\lambda}_3 = 0.1277$ and $\bar{\lambda}_4 = 0$. With the computed $K(s)$ and γ_c, the roots of: $\sigma_1(s)$ are $-0.4485 \pm 0.9825i$, -5.1029, $\sigma_2(s)$ are $-0.4689 \pm 0.6848i$, -5.0623, $\sigma_3(s)$ are -0.2977, -0.6834, -5.0189 and $\sigma_4(s)$ are 0, -1, -5. Since the roots of polynomials $\sigma_l(s)$, for $l = 1,2,3$, are in $\mathbb{C}^-$, they are stable.

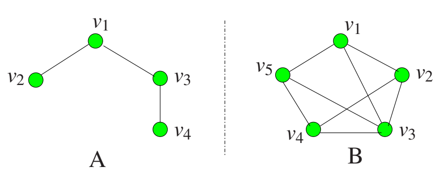

Figure 2: The network graphs $\mathcal{G}$ for the MASs with: i) four agents (A) and ii) five agents (B). The weights a_{ij} of the edges are considered to be one in both graphs.

The designed controller $K(s)$ is represented in the form (2), where $A_k = 0$, $b_k = 1$ and $c_k = 1$ (obtained using $tf2ss$ command in MATLAB). It is observed that the eigenvalues of A_{cl}, defined in the proof of Theorem 2, are equal to the union of the roots of $\sigma_i(s)$, for $i = 1,2,\cdots,4$. To obtain the

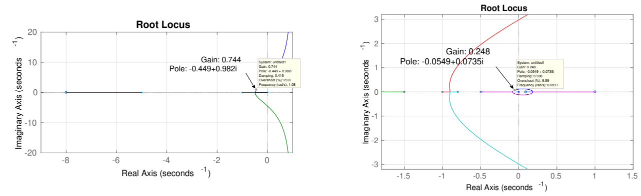

Figure 3: Root-locus of unity feedback systems for Example 1 (left side) and Example 2 (right side).

system matrices as in (9) and (11), following non-singular transformation matrices: $P = \left[\begin{smallmatrix} -0.5238 & 0.3810 \\ 0.1905 & -0.0476 \end{smallmatrix}\right]$ and $Q = 1$, are used. According to the proof of Theorem 2, the vector $\hat{h}_{cl}$ is constructed, which is: $\hat{h}_{cl} = [0\ 0.2\ 0\ 0.2\ 0\ 0.2\ 0\ 0.2\ 1\ 1\ 1\ 1]^T$. Further, using P and Q, the matrix M, as defined in the proof of Theorem 2, is constructed. Using M, we obtained:

$$h_{cl} = [0.2760 \quad 0.3795 \quad 0.2760 \quad 0.3795 \quad 0.2760 \quad 0.3795 \quad 0.2760 \quad 0.3795 \quad 0.1725 \quad 0.1725 \quad 0.1725 \quad 0.1725]^T,$$

which is indeed a right eigenvector of A_{cl} corresponding to the 0 eigenvalue. Then, corresponding left eigenvector is: $g_{cl} =$ [zeros(1,8) 1.4491 1.4491 1.4491 1.4491]. The initial conditions for individual agents are chosen as: $x_1(0) = [-1 \quad 5]^T$, $x_2(0) = [10 \quad 3]^T$, $x_3(0) = [-4 \quad 3]^T$ and $x_4(0) = [-10 \quad 10]^T$. Further, for controller: $\bar{x}_1(0) = 1$, $\bar{x}_2(0) = -2$, $\bar{x}_3(0) = 3$ and $\bar{x}_4(0) = 8$. Then, following the discussion in the proof of Theorem 2, we computed $\rho = 4$ and $x^* = [4 \quad 5.5]^T$. With the chosen initial conditions, the closed loop MAS system is simulated in MATLAB Simulink (version 2017a), and the output and state responses are depicted in Figure 4. It is observed that the outputs and the states of the agents converge to ρ and x^*, respectively. Hence, the consensus among the outputs and states of the agents are achieved.

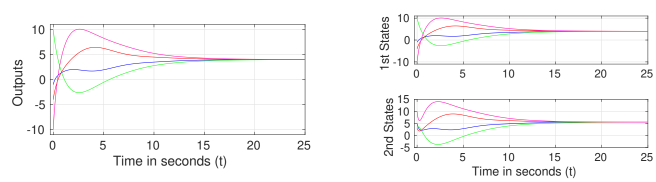

Figure 4: Consensus among the outputs (left side) and states (right side) of MAS in Example 1.

Example 2: In this example, we consider a MAS, consisting of a group of five identical agents, as in [20, Example 1]. The communication link among the agents is as per the graph in Figure 2-B (the undirected version of the directed graph in [20, Example 1] is considered). The dynamics of each agent is described by (1), where (the original state space model given in [20, Example 1] is made stable using a feedback gain vector $K = [24.4693 \ -15.8123 \ -8.0353]$):

$$A_p = \begin{bmatrix} 19.5777 & -12.8047 & -6.3402 \\ 9.9920 & -6.4180 & -3.1232 \\ 48.8038 & -31.8089 & -15.9597 \end{bmatrix}, \quad b_p = \begin{bmatrix} 0.8 \\ 0.4 \\ 2 \end{bmatrix}, \quad c_p = \begin{bmatrix} 5.32 \\ 1.56 \\ -2.44 \end{bmatrix}^T.$$

Corresponding to the matrices (A_p, b_p, c_p), the agent transfer function: $P(s) = \frac{0.9993s - 0.08996}{s^3 + 2.8s^2 + 2.35s + 0.6}$, which is stable, and satisfies the assumptions of Theorem 1. Since $P(s)$ has only one zero on the real axis of $\mathbb{C}^+$ (at 0.09), according to Theorem 1, we choose $z_{c_1} = 1$ and $p_{c_1} = -1$. Hence, the controller $K(s) = \frac{s-1}{s(s+1)}$. Using $P(s)$ and $K(s)$, the root-locus of the unity feedback system in Figure 1 is plotted, and it is shown in Figure 3. From the root-locus, $\overline{\kappa}$ is chosen as 0.248 according to Theorem 1. Since the largest eigenvalue of Laplacian matrix L is $\lambda_{max} = 5$, we have: $\gamma_c = 0.0496$. With the designed $K(s)$ and γ_c, the closed loop MAS is simulated in MATLAB Simulink. The output and state responses are depicted in Figure 5, and it is observed that consensus among the outputs and the states of the agents are achieved.

In [20, Example 1], the state consensus among the agents is achieved with an observer type distributed output feedback control. Hence, the order of the local controllers is equal to the order of agent, that is, three. On the other-hand, we achieve output as well as state consensus (by making the agent dynamics stable) with a second order local controller in the distributed output feedback control. In addition, our controller synthesis procedure is very simple, where classical control design approach is used, whereas the controller in [20, Example 1] is obtained via solving a set of linear matrix inequalities (LMI), for which, LMI solvers are required. In both the approaches, the interaction between the local controllers is not needed.

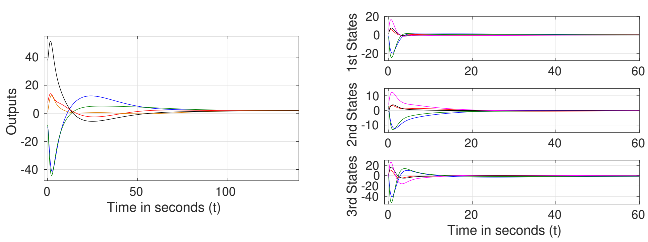

Figure 5: Consensus among the outputs (left side) and states (right side) of MAS in Example 2.

IV. Conclusion

In this work, we have proposed a novel feedback control architecture for a MAS to achieve consensus among the outputs and states of the agents. The control architecture has two components, one is a local controller for an individual agent and another is network gain. The proposed controller synthesis procedure is very simple. Further, the order of the local controller is much smaller than the order of the agent, and hence, it is easy to implement in the applications. In addition, there is no need of interaction between the local controllers, which helps in reducing the cost of the communication network. We obtained satisfactory results by implementing the proposed control architecture in the numerical examples.

It would be interesting to extend the proposed approach for multi-input multi-output systems. For this, the major challenges are associated with the controller synthesis procedure, where one has to deal with transfer function matrices, and network gain selection, where the concept of root-locus needs to be extended for transfer function matrices. We leave the detailed investigations in these directions for future research.

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