Small Signal Stability Criteria for Descriptor Form Power Network Model

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ABSTRACT

This work considers the problem of small signal stability of a lossless power network by representing its mathematical model in descriptor form. A stability criterion is derived by constructing two orthogonal projectors corresponding to the system model. These projectors are also used in: i) constructing a Lyapunov function, and ii) deriving another criterion to identify the presence of poles (finite) of the system to the right of a given vertical line in the complex plane. The latter criterion is required to analyze whether the dominant low frequency electro-mechanical modes, that cause oscillatory instability in the power network, settle down within a specified time limit. Using the derived criteria, linear matrix inequality feasibility problems are formulated to analyze small signal instability and oscillatory instability. Power network examples, including an IEEE benchmark system, are considered to demonstrate the developed approach.

KEYWORDS
Linear systems, descriptor systems, power networks, stability

1. Introduction

A power network, spreading over the globe, is vulnerable to a wide range of physical disturbances. The small signal disturbances, such as variations in the loads and generation, produce oscillatory instability in the power network (PN) (Kundur, 1994; Sauer & Pai, 1998). To analyze such small signal instability, traditionally, the original non-linear differential-algebraic equation (DAE) model of a PN is linearized around a steady state operating point. The resulting linearized DAE model, referred to as *descriptor form model*, is transformed into an ordinary state space (OSS) model by first expressing the algebraic variables in terms of the remaining variables, and then eliminating them from the system model (see the procedure discussed in (Sauer & Pai, 1998)). The stability is then inferred by computing the eigenvalues of the matrix associated with transformed OSS model.

In the process of transforming the original descriptor form power network (DFPN) model into an OSS model, a necessary step is to take the inverse of a Jacobian matrix (Kundur, 1994; Sauer & Pai, 1998). If the Jacobian matrix is ill-conditioned, then one may expect numerical errors in the transformed OSS model, which may also be reflected in the eigenvalues of the associated matrix. Some of the factors that may cause to produce ill-conditioned Jacobian ma-

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trix are: i) position of the slack bus, ii) existence of negative line reactance and iii) existence of certain type of radial architecture network (see (Tripathy, Prasad, Malik, & Hope, 1982) for details). When the DFPN model is reduced to an OSS model, the system matrices lose their sparsity structure, and hence, the performance of the algorithms for eigenvalues computation degrades (Rommes & Martins, 2009). Furthermore, since in the transformed OSS model the algebraic constraints are not available any more, the system may drift off from the algebraic constraints, without being noticed (Brenan, Campbell, & Petzold, 1989). To avoid such numerical difficulties, it is recommended in (Chu & Mehrmann, 1999; Rommes & Martins, 2009) that one should work with the original descriptor form model without transforming it into an OSS model. With these motivations, recently, several researchers, such as (Gross, Trenn, & Wirsen, 2014; Pasqualetti, Bicchi, & Bullo, 2011; Rommes & Martins, 2009; Rommes, Martins, & Freitas, 2010; Scholtz, 2004) have considered DFPN model for analyzing system theoretic properties and control design. In this work, we also preserve the original descriptor form model of the PN for deriving stability criteria.

The contributions of this work are as follows. A new small signal stability criterion is derived for a PN by representing its mathematical model in descriptor form. The derived criterion uses two orthogonal projectors which are constructed from the system matrices. With those projectors, i) a Lyapunov function is constructed and ii) another criterion is derived by which it can be verified if all the poles (finite) of a PN confine to the left of a given vertical line in the complex plane. In an interconnected PN, rather than just checking stability, it is important to analyze the minimum decay rate (settling time) of dominant low frequency electro-mechanical modes such as inter-area modes (0.1 Hz to 0.8 Hz). If those modes do not settle down within a specified time limit (usually 10 to 20 seconds), after clearing the fault, the oscillation may increase slowly, leading to oscillatory instability in the PN (Kundur, 1994; Sauer & Pai, 1998). The latter criterion is used to address this situation. Finally, to verify small signal stability of the system and minimum decay rate of the associated modes, linear matrix inequality (LMI) feasibility problems are formulated using the developed criteria. LMI feasibility problems are numerically tractable, and can be solved efficiently with interior-point optimization algorithms (Chilali & Gahinet, 1996).

The study of small signal stability of a PN is rich in the literature where several algorithms have been proposed to compute the eigenvalues of a matrix associated with the transformed OSS model (Kundur, 1994). It has been noticed that computing the eigenvalues of a matrix associated with a large-scale PN is often cumbersome, and hence, special numerical techniques have been proposed in (Kundur, 1994; Luo, 2011). Furthermore, since the oscillatory instability in an interconnected PN is primarily influenced by the presence of a few poorly damped low frequency electro-mechanical modes, special numerical techniques are proposed to identify the position of only those poles in the complex plane (Rommes et al., 2010). Another alternative approach to check if all the eigenvalues of a matrix lie in the open left half of the complex plane, is to solve the Lyapunov equation (Kundur, 1994). In this approach, one has to solve a set of linear equations and test if the resulting solution is positive definite. Similar approach is followed in our work to analyze small signal stability in a PN. Although the small signal stability analysis of a PN is well studied in the literature, to the best of our knowledge, such analysis is barely addressed (Rommes & Martins, 2009; Rommes et al., 2010) when the dynamic behavior of a PN is represented in descriptor form. Stability criteria for descriptor systems are presented in (Duan, 2010; Stykel, 2002). However, with those criteria it is not possible to verify the confinement of poles of a DFPN model to the left of a given vertical line in the complex plane, which is important in analyzing the oscillatory instability in a PN. The second criterion developed in this work particularly deals with such situations.

The remaining part of this article is organized as follows. Linearized structure preserving classical model of a PN is presented in Section 2. Since, the obtained linearized model is in descriptor from, some mathematical preliminaries on descriptor systems are presented to the end of Section 2. In Section 3, the stability criteria are derived, and a Lyapunov function is constructed. The concluding remarks are presented in Section 5, following to the demonstration of derived results with PN examples in Section 4.

Notation: The notation $Re(\lambda)$ is used to denote the real part of a complex number λ. $A \succ 0$ denotes that A is a symmetric positive definite (SPD) matrix, and $\det(\bullet)$ denotes the determinant of a matrix. The identity matrix is denoted as I.

2. Preliminaries

Consider an $n+m$ bus PN with n generator buses, indexed as: $1, 2, \cdots, n$, and m load buses, indexed as: $n+1, n+2, \cdots, n+m$. Let δ_i, ω_i, M_i, D_i, X'_{d_i} be the rotor angle, rotor speed, inertia constant, damping coefficient, transient reactance of the i^{th} generating unit, respectively, and P_{M_i} be the mechanical power input to the i^{th} generating unit. Further, denote θ_i and P_i as the phase angle and the active power injected at i^{th} bus. Let Y_N be the bus admittance matrix of the network with entries $Y_{ik} := G_{ik} + jB_{ik}$. Assume that the transmission lines are lossless, that is, $G_{ik} = 0$, and the bus voltages are close to their nominal rated values, that is, 1 pu (per unit). Then, linearized model[1] of the PN can be represented as (see also (Gross et al., 2014; Scholtz, 2004)):

$$\dot{\delta}_i = \omega_i, \quad M_i \dot{\omega}_i = P_{M_i} - D_i \omega_i - \frac{1}{X'_{d_i}}(\delta_i - \theta_i), \tag{1a}$$

$$P_k = -\frac{1}{X'_{d_k}}(\delta_k - \theta_k) + \sum_{j=1}^{m} B_{kj}(\theta_k - \theta_j), \text{ for } k = 1, 2, \cdots, n, \tag{1b}$$

$$P_k = \sum_{j=1}^{m} B_{kj}(\theta_k - \theta_j), \text{ for } k = n+1, n+2, \cdots, n+m. \tag{1c}$$

Considering the network parameters B_{ik}, let us define a matrix L as in (31) (see the Appendix), and write it as: $L = \left[\begin{array}{c:c} \mathbf{B_{11}} & \mathbf{B_{12}} \\ \hdashline \mathbf{B_{21}} & \mathbf{B_{22}} \end{array}\right]$. Further, define the following variables:

$$\begin{aligned}
&\delta := [\delta_1,\ \delta_2,\ \cdots,\ \delta_n]^T, \ \ \omega := [\omega_1,\ \omega_2,\ \cdots,\ \omega_n]^T, \\
&\theta_g := [\theta_1,\ \theta_2,\ \cdots,\ \theta_n]^T, \ \ \theta_l := [\theta_{n+1},\ \theta_{n+2},\ \cdots,\ \theta_{n+m}]^T \\
&P_M := [P_{M_1}\ P_{M_2}\ \cdots\ P_{M_n}]^T, \ P_g := [P_1\ P_2\ \cdots\ P_n]^T, \\
&P_l := [P_{n+1}\ P_{n+2}\ \cdots\ P_{n+m}]^T, \ \mathbf{M} := \mathrm{diag}\{M_1, M_2. \cdots, M_n\}, \\
&\mathbf{D} := \mathrm{diag}\{D_1, D_2. \cdots, D_n\}, \ \mathbf{X_d} := \mathrm{diag}\{X'_{d_1}, X'_{d_2}, \cdots, X'_{d_n}\},
\end{aligned}$$

where $\mathrm{diag}\{\bullet\}$ is a diagonal matrix. Then, (1) can compactly be written as:

$$E\dot{x} = Ax + Bu \tag{2}$$

[1] We have considered the structure-preserving classical model of a PN where a generator is represented with constant magnitude voltage source behind a transient reactance (Sauer & Pai, 1998, Chapter 7).

where

$$E=\left[\begin{array}{cc:cc} I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right],\ A=\left[\begin{array}{cc:cc} \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ -\mathbf{X_d^{-1}} & -\mathbf{D} & \mathbf{X_d^{-1}} & \mathbf{0} \\ \hdashline -\mathbf{X_d^{-1}} & \mathbf{0} & \mathbf{X_d^{-1}}+\mathbf{B_{11}} & \mathbf{B_{12}} \\ \mathbf{0} & \mathbf{0} & \mathbf{B_{21}} & \mathbf{B_{22}} \end{array}\right],$$

$$B=\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -I \end{bmatrix},\quad x=\begin{bmatrix} \delta \\ \omega \\ \theta_g \\ \theta_l \end{bmatrix},\quad u=\begin{bmatrix} P_M \\ P_g \\ P_l \end{bmatrix}. \tag{3}$$

Note that the linearized model (2) is in the descriptor form. Hence, it is important to ensure that (2) is *regular*, that is, (2) is solvable for all input u and its solution is uniquely determined by a given initial condition $x(0)=x_0$ (Dai, 1989). It is shown in (Gross et al., 2014) that the descriptor form model (2) is regular, provided every load bus is connected to at least one generator bus. Under this condition, (2) satisfies the following relation: $\det(sE-A)\neq 0$ where $s\in\mathbb{C}$ (Dai, 1989). In the remaining part of this article it is assumed that every load bus in the PN is connected to at least one of the generator buses (a path exists from load bus to a generator bus). Hence, (2) will satisfy: $\det(sE-A)\neq 0$. The roots: $\{\lambda_1,\lambda_2,\cdots,\lambda_d\}$ of the polynomial: $\det(sE-A)$ are referred to as *finite poles* of (2) (Dai, 1989). The PN is said to be *small signal stable* if all the finite poles of (2) lie in the open left half of the complex plane, that is, $Re(\lambda_i)<0$ for $i=1,2,\cdots,d$ (see the definition of *stability* for a descriptor system in (Dai, 1989; Duan, 2010)). Further, the PN is said to be *small signal γ-stable* if the finite poles λ_i's of (2) satisfy:

$$Re(\lambda_i)<-\gamma,\ \text{ for } i=1,2,\cdots,d,\ \text{ with } \gamma>0. \tag{4}$$

Using the above definitions, stability criteria for (2) are derived in the next section.

3. Main Results

Let the generator buses in the PN be indexed such that the inertia constants M_i, for $i=1,2,\cdots,n$, satisfy:

$$M_1\geq M_2\geq\cdots\geq M_n>0. \tag{5}$$

Note that each M_i also satisfies: $0<M_i<1$ (Kundur, 1994; Sauer & Pai, 1998). It is then clear form the theory of singular value decomposition (Golub & Van Loan, 1996) that the orthogonal matrices U and V, required to compute the decomposition:

$$U^TEV=\begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \text{ where } \Sigma=\begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \tag{6}$$

are the identity matrices. Corresponding to system (2), let us define the following projector matrices:

$$P_d:=E^+E,\quad\text{and}\quad P_d':=I-E^+E, \tag{7}$$

where E^+ is the Moore-Penrose inverse of E (Golub & Van Loan, 1996). Then, the following result holds.

Theorem 3.1. *Consider the linearized PN model* (2)*. Assume that* (5) *holds. Let the projector matrices be as in* (7)*. Define a matrix*

$$Z_d := P_d - (P_d' A P_d')^+ (P_d' A P_d). \tag{8}$$

Then, the finite poles of (2) *will satisfy: $Re(\lambda_i) < 0$, for $i = 1, 2, \cdots, d$, if and only if for a given SPD matrix Q, there exists a SPD matrix X which satisfies:*

$$X E^+ A Z_d + Z_d^T A^T (E^+)^T X = -P_d Q P_d. \tag{9}$$

Proof. *If part*: Assume that for a given $Q \succ 0$, there exists $X \succ 0$ which satisfies (9). Since (5) holds, the orthogonal matrices U and V in (6) are identity matrices. Hence,

$$E^+ = \begin{bmatrix} \Sigma^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

The projector matrices, defined in (7), are then:

$$P_d = E^+ E = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \text{ and } P_d' = I - E^+ E = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}. \tag{10}$$

Let us partition the matrix A in (3), conformably with $\begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ where Σ is as in (6), and write it as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{11}$$

It is mentioned in (Gross et al., 2014) that the linearized PN model (2) is an *index one* descriptor system. Further, since the orthogonal matrices U and V in (6) are identity matrices, it follows from (Duan, 2010, Chapter 7) that the matrix A_{22} is invertible. Then, it can be shown that

$$Z_d = P_d - (P_d' A P_d')^+ (P_d' A P_d) = \begin{bmatrix} I & \mathbf{0} \\ -A_{22}^{-1} A_{21} & \mathbf{0} \end{bmatrix}.$$

Let us define the symmetric matrices X and Q as follows:

$$X := \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}, \quad Q := \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}. \tag{12}$$

Then,

$$X E^+ A Z_d = \begin{bmatrix} X_{11} A_d & \mathbf{0} \\ X_{12}^T A_d & \mathbf{0} \end{bmatrix}, \text{ where } A_d := \Sigma^{-1} \left(A_{11} - A_{12} A_{22}^{-1} A_{21} \right). \tag{13}$$

Furthermore,

$$XE^{+}AZ_d + Z_d^T A^T (E^{+})^T X = \begin{bmatrix} A_d^T X_{11} + X_{11} A_d & A_d^T X_{12} \\ X_{12}^T A_d & \mathbf{0} \end{bmatrix}. \quad (14)$$

Using (10) and (12),

$$P_d Q P_d = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} Q_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \quad (15)$$

Using (14) and (15), the relation (9) yields

$$A_d^T X_{11} + X_{11} A_d = -Q_{11} \quad (16)$$

$$A_d^T X_{12} = \mathbf{0}. \quad (17)$$

Since $X \succ 0$ satisfies (9) for a given $Q \succ 0$, the matrices Q_{11} and X_{11} will also satisfy (16). Furthermore, since $Q_{11} \succ 0$ and $X_{11} \succ 0$, it follows from the Lyapunov theory for OSS that all the eigenvalues of A_d have negative real part (Kailath, 1980; Khalil, 1996). We have shown in Proposition 6.1 (see the Appendix) that the eigenvalues of A_d are equal to the finite poles of (2). Hence, $Re(\lambda_i) < 0$, for $i = 1, 2, \cdots, d$.

Only if part: Assume that the finite poles of (2) satisfy: $Re(\lambda_i) < 0$, for $i = 1, 2, \cdots, d$. According to Proposition 6.1, λ_i's are also the eigenvalues of A_d. Hence, according to the Lyapunov theory, for every choice of $Q_{11} \succ 0$, a unique $X_{11} \succ 0$, satisfying (16), can be obtained (Khalil, 1996). Furthermore, by setting $X_{12} = \mathbf{0}$, the relation (17) will also be satisfied. Since there is no constraint on X_{22}, one can choose it to be a SPD matrix. Hence, for a given $Q \succ 0$, the solution X, satisfying (9), is: $X = \begin{bmatrix} X_{11} & \mathbf{0} \\ \mathbf{0} & X_{22} \end{bmatrix}$, with $X_{11} \succ 0$ and $X_{22} \succ 0$. This completes the proof. □

According to Theorem 3.1, the small signal stability of a PN can be verified by solving (9) and checking if $X \succ 0$ for a given $Q \succ 0$. Note that the choice of Q is arbitrary in the set of SPD matrices, and is not restricted to a particular one. Since (9) is a linear matrix equation with free variable X, the stability can be inferred by formulating the following feasibility problem.

Problem 3.2. *For a given $Q \succ 0$ (one can choose Q to be an identity matrix), find $X \succ 0$ such that*

$$\text{(i)}\ \ XE^{+}AZ_d + Z_d^T A^T (E^{+})^T X = -P_d Q P_d.$$

A feasible solution to Problem 3.2 ensures that $Re(\lambda_i) < 0$, for $i = 1, 2, \cdots, d$, and hence, the PN is small signal stable. Note that, constraint (i) in Problem 3.2 is linear in X, and hence, can be solved with an LMI solver, such as *SeDuMi* (Peaucelle, Henrion, & Labit, 2002). The stability property of a DFPN model can also be established by constructing a suitable Lyapunov function (Khalil, 1996). For completeness, a Lyapunov function for (2) using Theorem 3.1 is constructed next.

Theorem 3.3. *Consider the DFPN model* (2)*, and assume that input $u(t) \equiv 0$, for $t > 0$. Let the projector matrix P_d be as in* (7)*. Let $X \succ 0$ be the solution of* (9) *for a given SPD matrix*

Q. Then,

$$\mathscr{V}(x) := x^T P_d X P_d x, \tag{18}$$

is a Lyapunov function for:

$$E\dot{x} = Ax. \tag{19}$$

Proof. The function $\mathscr{V}(x)$ will be a Lyapunov function if the following conditions hold (Khalil, 1996):

$$\text{(i) } \mathscr{V}(0) = 0, \ \ \text{(ii) } \mathscr{V}(x) > 0, \ \ \forall\, P_d x \neq 0, \ \ \text{(iii) } \dot{\mathscr{V}}(x) < 0, \ \ \forall\, P_d x \neq 0,$$

where $\dot{\mathscr{V}}(x)$ is the derivative of $\mathscr{V}(x)$ along the solution trajectories of (19). Since $X \succ 0$, it can easily be shown that $\mathscr{V}(x)$ satisfies conditions (i) and (ii). To show that $\mathscr{V}(x)$ also satisfies (iii), let us take the derivative of $\mathscr{V}(x)$ along the trajectories of (19). Then,

$$\dot{\mathscr{V}}(x) = \dot{x}^T P_d X P_d x + x^T P_d X P_d \dot{x}.$$

Pre-multiply E^+ on both sides of (19). Then, $P_d \dot{x} = E^+ A x$, and hence

$$\dot{\mathscr{V}}(x) = x^T A^T (E^+)^T X P_d x + x^T P_d X E^+ A x. \tag{20}$$

Let us define the vectors x_d and x_a as follows: $x_d := P_d x,\ x_a := P_d' x$ (Baum, 2017). It is then clear that $x = x_d + x_a$. By replacing x with $x_d + x_a$, and $P_d x$ with x_d in (20),

$$\dot{\mathscr{V}}(x) = x_d^T \left[A^T (E^+)^T X + X E^+ A\right] x_d + x_a^T A^T (E^+)^T X x_d + x_d^T X E^+ A x_a. \tag{21}$$

Pre-multiply P_d' on both sides of (19). Then, $P_d' A x = 0$, which implies $P_d' A x_a + P_d' A x_d = 0$. Further, since $P_d' x_a = x_a$ and $P_d x_d = x_d$,

$$P_d' A P_d' x_a = -P_d' A P_d x_d \tag{22}$$

Pre-multiplying $(P_d' A P_d')^+$ on both sides of (22),

$$(P_d' A P_d')^+ (P_d' A P_d') x_a = -(P_d' A P_d')^+ (P_d' A P_d) x_d. \tag{23}$$

Note that $(P_d' A P_d')^+ (P_d' A P_d') x_a = P_d' x_a = x_a$. Hence, (23) becomes:

$$x_a = -(P_d' A P_d')^+ (P_d' A P_d) x_d. \tag{24}$$

Using the definition of Z_d as in (8), (24) becomes: $x_a = (Z_d - P_d) x_d$. Further, using the relation $P_d x_d = x_d$, and substituting x_a with $(Z_d - P_d) x_d$ in (21):

$$\dot{\mathscr{V}}(x) = x_d^T \left[X E^+ A Z_d + Z_d^T A^T (E^+)^T X\right] x_d. \tag{25}$$

Since $X \succ 0$ satisfies (9) for a given $Q \succ 0$, (25) becomes:

$$\dot{\mathscr{V}}(x) = -x_d^T P_d Q P_d x_d = -x_d^T Q x_d.$$

Since $Q \succ 0$, $\dot{\mathscr{V}}(x)$ is negative for all $P_d x = x_d \neq 0$. Hence $\mathscr{V}(x)$ is a Lyapunov function for (19). This completes the proof. $\square$

As it is discussed in the introduction that in addition to the stability analysis of a PN, it is often essential to verify if the dominant electro-mechanical modes, following to a fault, settle down within a specified time limit (to avoid oscillatory instability). If λ_i's satisfy (4), then the associated modes (exponential terms that appear in the state response) will settle down within $t_s = \frac{4}{\gamma}$ seconds (Chilali & Gahinet, 1996). Hence, the oscillatory instability in the PN can be avoided by ensuring that all the finite poles of (2) belong to the left of a vertical line in the complex plane, chosen suitably according to the time-frame of interest. In the next result we propose another criterion where a feasible solution guarantees that (4) holds.

Theorem 3.4. *Consider the linearized power network model* (2)*. Let the projector P_d be as in* (7)*, and Z_d be as in* (8)*. Then, the finite poles of* (2) *will satisfy* (4)*, if and only if for a given SPD matrix Q, there exists a SPD matrix X which satisfies:*

$$XE^{+}AZ_d + Z_d^T A^T (E^{+})^T X + 2\gamma P_d X P_d = -P_d Q P_d. \tag{26}$$

Proof. *If part:* Assume that the matrices X and Q are as in (12). Then, using (14), and the fact that

$$2\gamma P_d X P_d = \begin{bmatrix} 2\gamma X_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

the relation (26) can be represented as:

$$\begin{bmatrix} A_d^T X_{11} + X_{11} A_d + 2\gamma X_{11} & A_d^T X_{12} \\ X_{12}^T A_d & \mathbf{0} \end{bmatrix} = -\begin{bmatrix} Q_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

This leads to the following set of matrix equations:

$$A_d^T X_{11} + X_{11} A_d + 2\gamma X_{11} = -Q_{11} \tag{27}$$

$$A_d^T X_{12} = \mathbf{0}. \tag{28}$$

For a given $Q \succ 0$, the matrix $X \succ 0$ satisfies (26). Hence, the matrices Q_{11} and X_{11} will satisfy (27). Further, since $Q_{11} \succ 0$ and $X_{11} \succ 0$, it follows from (Chilali & Gahinet, 1996) that all the eigenvalues of A_d, and hence, the finite poles of (2), satisfy (4).

Only if part: Assume that (4) holds. Then, it follows from (Chilali & Gahinet, 1996) that for every choice of $Q_{11} \succ 0$, there exists $X_{11} \succ 0$ which satisfies (27). By setting $X_{12} = \mathbf{0}$, the relation (28) will also hold. Since there is no constant on X_{22}, it can be chosen as a SPD matrix. This shows the existence of SPD matrix X. The proof is completed. $\square$

We formulate the following LMI feasibility problem to verify if all the finite poles of (2) lie in the left of a vertical line at $-\gamma$ in the complex plane.

Problem 3.5. *For a given $Q \succ 0$, find $X \succ 0$ such that*

$$(i)\ \ XE^{+}AZ_d + Z_d^T A^T (E^{+})^T X + 2\gamma P_d X P_d = -P_d Q P_d.$$

A feasible solution to Problem 3.5 ensures that (4) holds, and hence the PN is small signal γ-stable.

Remark 1. In the DFPN model (2), a finite pole at 0 is expected due to the non-uniqueness of absolute rotor angle δ_i (Kundur, 1994; Sauer & Pai, 1998). Hence, to analyze the small signal stability with the proposed criteria, it is required to modify the linearized model (2) by choosing one of the machine as reference, and expressing the rotor angle of other machines with respect to the reference (more details can be found in the next section, and in (Sauer & Pai, 1998, Chapter 8)).

4. Illustrative Examples

Example 1: A 2-generator, 3-bus PN, as shown in Figure 1, is considered where the system frequency $f = 50$ Hz (Hertz). The machine data are: $M_1 = 0.0407$ secs^2/rad (seconds^2/radian), $M_2 = 0.0192$ secs^2/rad, $D_1 = 0.0081$ pu, $D_2 = 0.0057$ pu, $X'_{d_1} = 0.1198$ pu and $X'_{d_2} = 0.1813$ pu. The network parameters ($Y_{ik} = G_{ik} + jB_{ik}$ where Y_{ik} is an element of bus admittance matrix Y_N) are considered as follows (it is assumed that transmission lines are lossless, that is, $G_{ik} = 0$): $B_{12} = 9.7840$ pu, $B_{13} = 5.9760$ pu, and $B_{23} = 5.5880$ pu. Since the load bus: 3, as shown in Figure 1, is connected to the generator buses: 1,2, it follows from Section 2 that the DFPN model is regular. The system matrices E and A are obtained according to the procedure discussed in Section 2. The computed finite poles are: $-0.1340 \pm j14.2089$, -0.2320 and 0 (obtained by computing the eigenvalues of A_d as in (13)).

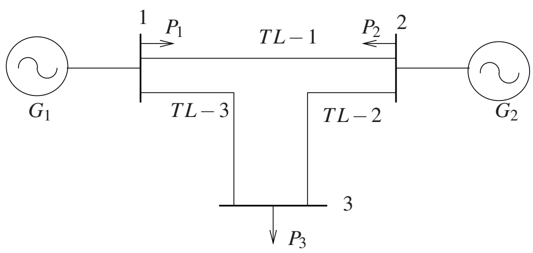

Figure 1. A 2-generator, 3-bus power network with synchronous generators G_1 and G_2, generator buses: 1 and 2, and load bus: 3. Transmission lines are represented as $TL-1$, $TL-2$ and $TL-3$. Power delivered to the loads at bus 1, bus 2 and bus 3 are represented as P_1, P_2 and P_3, respectively.

It is discussed in Remark 1 that the presence of a finite pole at 0 is due to the non-uniqueness of absolute rotor angle of the generators. Hence, to analyze small signal stability (rotor angle stability), generator G_1, shown in Figure 1, is chosen as reference, and expressed the rotor angle of generator G_2 with respect to G_1. With this modification, following quantities are defined:

$$\delta_i' := \delta_i - \delta_1,\ \dot{\delta_i}' := \omega_i - \omega_1,\ \theta_i' := \theta_i - \delta_1,\ \text{ for } i = 1,2.$$

Since $\delta_1' = \dot{\delta_1}' = 0$, the differential equation corresponding to δ_1 is deleted from (2). Then, with the new state variables δ_i', ω_i and θ_i', the system matrices are:

$$E' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0407 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0192 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$
A' = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -0.0081 & 0 & 8.3472 & 0 & 0 \\ -5.5157 & 0 & -0.0057 & 0 & 5.5157 & 0 \\ 0 & 0 & 0 & 24.1072 & -9.7840 & -5.9760 \\ -5.5157 & 0 & 0 & -9.7840 & 20.8877 & -5.5880 \\ 0 & 0 & 0 & -5.9760 & -5.5880 & 11.5640 \end{bmatrix}.
$$

Now, the finite poles are $-0.1340 \pm j14.2089$ and -0.2320. Since all the finite poles are in the open left half of the complex plane, the PN is small signal stable. To verify this with the proposed criterion, Problem 3.2 is solved with *SeDuMi* by considering $Q = I$. Following feasible solution is obtained:

$$
X = \begin{bmatrix} 214.6449 & -0.1114 & -0.0487 & 0 & 0 & 0 \\ -0.1114 & 3.0568 & -0.1252 & 0 & 0 & 0 \\ -0.0487 & -0.1252 & 1.5042 & 0 & 0 & 0 \\ 0 & 0 & 0 & 82.8297 & 0 & 0 \\ 0 & 0 & 0 & 0 & 82.8297 & 0 \\ 0 & 0 & 0 & 0 & 0 & 82.8297 \end{bmatrix}.
$$

The feasibility of Problem 3.2 ensures that all the finite poles lie in the open left half of the complex plane. Hence, the PN is small signal stable, that is, following to a small signal disturbance in the PN, the relative rotor angle $(\delta_2(t) - \delta_1(t))$ will be finite as $t \to \infty$. Further, the speed difference $(\omega_2(t) - \omega_1(t))$ between two generators will converge to zero as the time advances, and the generators will remain in synchronism.

To verify the criterion proposed in Theorem 3.4, Problem 3.5 is solved by setting $Q = I$ and $\gamma = 0.1$. A feasible solution is obtained, which confirms that all of the finite poles are confined to the left of a vertical line at -0.1 in the complex plane. Hence, all of the modes will settle down within $t_s = 40$ secs. This conveys that if the two synchronous machines G_1 and G_2 are oscillating against each other, following to a disturbance, then they will come to a steady state within 40 secs. By increasing the damping coefficients D_i, that is, by choosing $D_1 = 0.0326$ pu and $D_2 = 0.0172$ pu, it is observed that the finite poles are confined to the left of a vertical line at -0.4. Hence, the oscillation will reach to a steady state within 10 secs.

The performance of the developed stability criterion is verified next at multiple operating points of the PN. Assume that the transmission lines $(TL-i)$ between the buses i and j, as depicted in Figure 1, are double circuit lines. If a fault occurs in one of the lines of the double circuit line between bus i and bus j, then that line needs to be removed from the PN by tripping circuit breakers available at both ends of the transmission line. Under this scenario the PN will switch to a different operating point. In the new operating condition the network parameter B_{ij} between bus i and bus j will become double, while others will remain same. Following four fault scenarios are considered in the transmission lines, and the stability is studied for each operating conditions.

(1) *Fault in one of the lines of TL-1*: The network parameters corresponding to this fault scenario are mentioned in Table 1. By computing the system matrices E' and A', Prob-

lem 3.2 is solved. Following feasible solution is obtained for $Q = I$:

$$X = \begin{bmatrix} 235.8129 & -0.1112 & -0.0490 & 0 & 0 & 0 \\ -0.1112 & 3.0559 & -0.1244 & 0 & 0 & 0 \\ -0.0490 & -0.1244 & 1.5034 & 0 & 0 & 0 \\ 0 & 0 & 0 & 91.2916 & 0 & 0 \\ 0 & 0 & 0 & 0 & 91.2916 & 0 \\ 0 & 0 & 0 & 0 & 0 & 91.2916 \end{bmatrix}.$$

Hence, the PN is stable. This is also clear from the finite poles of the system which are are mentioned in Table 1.

(2) *Faults in one of the lines of both TL-1 and TL-2*: The network parameters for the faulted system are given in Table 1. By setting $Q = I$, following feasible solution is obtained for Problem 3.2:

$$X = \begin{bmatrix} 237.1127 & -0.1112 & -0.0490 & 0 & 0 & 0 \\ -0.1112 & 3.0559 & -0.1244 & 0 & 0 & 0 \\ -0.0490 & -0.1244 & 1.5033 & 0 & 0 & 0 \\ 0 & 0 & 0 & 91.7806 & 0 & 0 \\ 0 & 0 & 0 & 0 & 91.7806 & 0 \\ 0 & 0 & 0 & 0 & 0 & 91.7806 \end{bmatrix}.$$

Hence, the PN is stable. The finite poles of the faulted system are given in Table 1.

(3) *Fault in one of the lines of TL-3*: The network parameters under this fault scenario are given in Table 1. By setting $Q = I$, following feasible solution is obtained for Problem 3.2:

$$X = \begin{bmatrix} 217.6785 & -0.1113 & -0.0488 & 0 & 0 & 0 \\ -0.1113 & 3.0566 & -0.1251 & 0 & 0 & 0 \\ -0.0488 & -0.1251 & 1.5041 & 0 & 0 & 0 \\ 0 & 0 & 0 & 83.9303 & -0.0635 & -0.0002 \\ 0 & 0 & 0 & -0.0635 & 84.1840 & 0.0011 \\ 0 & 0 & 0 & -0.0002 & 0.0011 & 84.2663 \end{bmatrix}.$$

Hence, the PN is stable under this fault condition. The finite poles of the new system are given in Table 1.

(4) *Faults in both lines of TL-1 and TL-2*: The network parameters for this fault scenario are mentioned in Table 1. No feasible solution is obtained by solving Problem 3.2, and hence, the PN is unstable under this fault condition. The finite poles are mentioned in Table 1. Observe that there is one finite pole at 0, and that causes the system unstable.

Table 1. Pole locations for 2-generator 3-bus system under different fault scenarios

Fault in transmission lines	Line parameters in pu	Finite poles
$TL-1$	$B_{12} = 19.5680$, $B_{13} = 5.9760$ $B_{23} = 5.5880$	-0.2320 $-0.1340 \pm 14.8989i$
$TL-1$ and $TL-2$	$B_{12} = 19.5680$, $B_{13} = 5.9760$ $B_{23} = 11.1760$	-0.2320 $-0.1340 \pm 14.9403i$
$TL-3$	$B_{12} = 9.7840$, $B_{13} = 11.9520$ $B_{23} = 5.5880$	-0.2320 $-0.1340 \pm 14.3098i$
$TL-1$ and $TL-2$	$B_{12} = 0$, $B_{13} = 5.9760$, $B_{23} = 0$	-0.2, -0.3, 0

Example 2: In this example, a 6-generator, 13-bus PN, as shown in Figure 2, is considered

Table 2. Machine Data for 6-generator, 13-bus PN

Generators	M_i (in secs^2/rad)	D_i (in pu)	X'_{d_i} (in pu)
$G1$	0.0451	0.0045	0.1500
$G2$	0.0424	0.0064	0.1800
$G3$	0.0398	0.0080	0.2000
$G4$	0.0371	0.0093	0.2500
$G5$	0.0345	0.0103	0.2500
$G6$	0.0318	0.0111	0.3000

(Tripathy et al., 1982) (the buses are renamed so as to match with the notations used in this article). The machine data are considered as in Table 2 with system frequency $f = 60$ Hz. The network parameters B_{ik} are computed by constructing the network bus admittance matrix Y_N from the data given in (Tripathy et al., 1982, Appendix II), and setting $G_{ik} = 0$ ($Y_{ik} = G_{ik} + jB_{ik}$ where Y_{ik} is an element of Y_N)[2]. With the computed parameters B_{ik}, it is observed that the PN model is regular. The linearized model is modified according to the procedure discussed in previous example, by considering machine one ($G1$ in Figure 2) as reference. With the modified data, a feasible solution to Problem 3.2 is obtained by setting $Q = I$. Hence, the PN model is stable, that is, the synchronous machines will remain in synchronism, following to a small signal disturbance. This is also confirmed by observing the computed finite poles: -0.2150, $-0.1185 \pm j6.6557$, $-0.1393 \pm j10.4453$, $-0.1154 \pm j7.8487$, $-0.0620 \pm j9.0576$, $-0.1323 \pm j8.9681$.

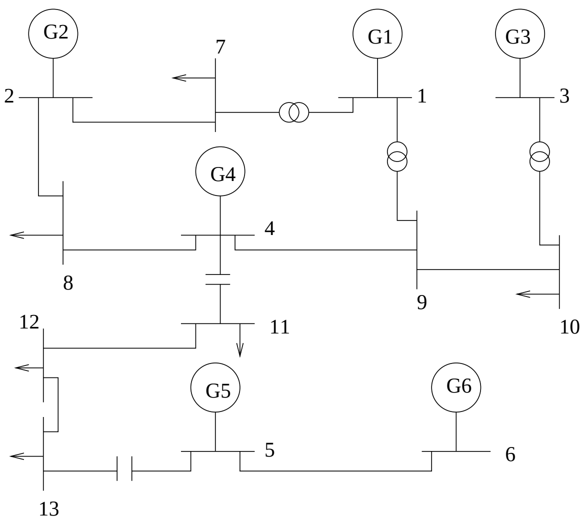

Figure 2. A 6-generator, 13-bus PN with generator buses: $1, 2, \cdots, 6$, and load buses: $7, 8, \cdots, 13$.

Example 3: This example considers a 10-generator, 39-bus New-England PN, as depicted in Figure 3 (Padiyar, 2008). This PN is considered to be an IEEE benchmark system (Canizares et al., 2017). The generator buses are reindexed such that the inertia constants $M_i = \frac{2H_i}{2\pi f}$, for $i = 1, 2, \cdots, 10$, will satisfy (5). Generator $G1$ is considered to be the reference. The machine data are given in Table 3, and the system frequency is considered as 60 Hz. The network parameters B_{ik} are computed by constructing the network bus admittance matrix Y_N

[2]The parameters G_{ik}'s are set to 0, since the linearized PN model (2) is obtained by assuming lossless transmission lines. Note that, the proposed stability criteria (9) and (26) can still be used for a linearized PN model when it is obtained without assuming lossless transmission lines.

Table 3. Machine Data for 10-generator, 39-bus PN

Generators	H_i (in secs^2)	M_i (in secs^2/rad)	D_i (in pu)	X'_{d_i} (in pu)
$G1$	500	2.6526	0.2653	0.0060
$G2$	42	0.2228	0.0334	0.0040
$G3$	35.8	0.1899	0.0342	0.0531
$G4$	34.8	0.1846	0.0369	0.0500
$G5$	34.5	0.1830	0.0403	0.0570
$G6$	30.3	0.1607	0.0402	0.0647
$G7$	28.6	0.1517	0.0425	0.0436
$G8$	26.4	0.1401	0.0420	0.0490
$G9$	26	0.1379	0.0441	0.0660
$G10$	24.3	0.1289	0.0451	0.0570

from the transmission line data given in (Padiyar, 2008, Appendix B), and setting $G_{ik} = 0$. The system matrices E and A are modified to obtain E' and A' according to the procedure discussed in Example-1. With the modified data, a feasible solution is obtained by solving Problem 3.2. Hence, the PN is stable. This also confirms from the location of finite poles of the system, which are: -0.1503, $-0.0949 \pm j11.2385$, $-0.0995 \pm j4.3162$, $-0.1455 \pm j9.7028$, $-0.1316 \pm j9.1932$, $-0.1160 \pm j6.5880$, $-0.1140 \pm j7.0733$, $-0.1526 \pm j8.4342$, $-0.1089 \pm j8.1675$ and $-0.1368 \pm j7.6271$.

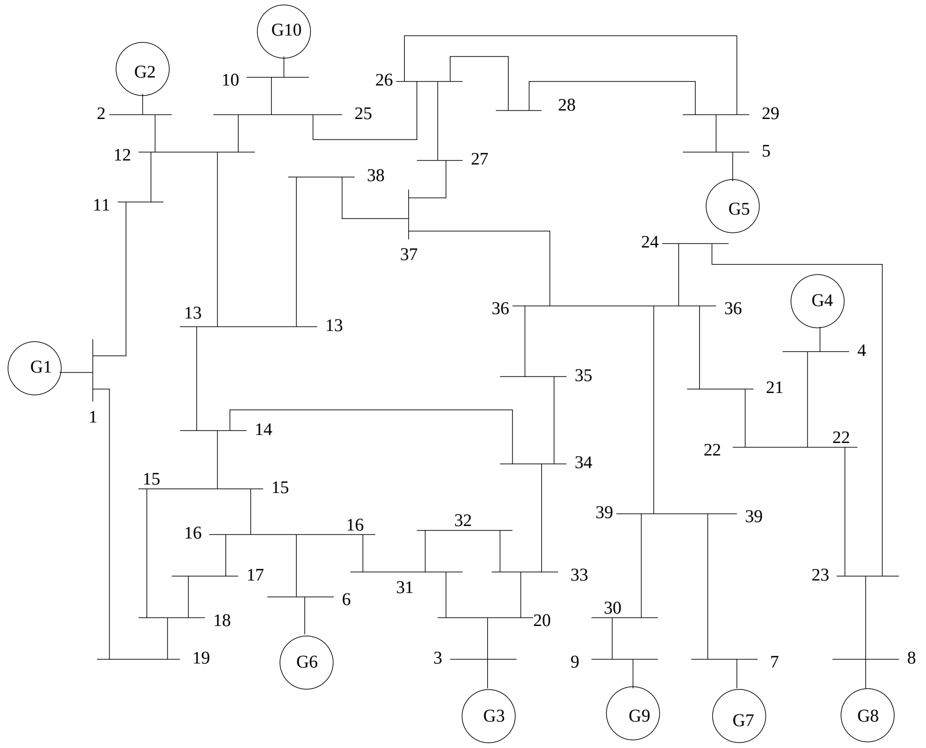

Figure 3. IEEE 39-bus PN with 10 generator buses, named as: $1, 2, \cdots, 10$, and 29 load buses, named as: $11, 12, \cdots, 39$.

5. Conclusion

In this work, two stability criteria are proposed for a DFPN model. The first criterion is useful to verify if all the finite poles of a DFPN model lying in the open left half of complex plane. An infeasible solution ensures that there is at least one finite pole which is lying in the (closed) right half of complex plane, and hence, the system is not small signal stable. On the other hand, the second criterion is to investigate if all the finite poles of a DFPN model lying to the left of a given vertical line at $-\gamma$ in the complex plane. This criterion is useful to analyze the oscillatory instability in the PN. If the criterion produces an infeasible solution for a given γ, then one may expect the presence of low frequency electro-mechanical modes which are not settling within the desired time. As a result, the PN may experience oscillatory instability following to a disturbance.

Another advantage of the developed criteria is that they are linear in unknown matrix variables. Hence, the existence of feasible solutions can be verified by solving the proposed LMI feasibility problems. LMI feasibility problems are numerically tractable and can be solved efficiently using existing solvers, such as MATLAB's *LMI Control Toolbox* and *SeDuMi*. One may also choose the algorithms proposed in (Miller, 2001; Miller & Smith, 2000) to solve efficiently the large order LMIs associated with large-scale PNs.

6. Appendix

Proposition 6.1. *The finite poles of* (2) *and the eigenvalues of* A_d*, as defined in* (13)*, are equal, that is,* $\Lambda\{det(sE-A)\} = \Lambda\{det(sI-A_d)\}$*, where* $\Lambda\{\bullet\}$ *denotes the roots of a polynomial.*

Proof. Using the decomposition (6) and (11), and identifying the fact that U and V are identity matrices, following relation holds:

$$\Lambda\{\det\ (sE-A)\} = \Lambda\left\{\det\left(\begin{bmatrix} s\Sigma - A_{11} & -A_{12} \\ -A_{21} & -A_{22}\end{bmatrix}\right)\right\}. \tag{29}$$

Pre-multiplying $\begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ \mathbf{0} & I \end{bmatrix}$ with the matrices in (29) leads to

$$\Lambda\{\det\ (sE-A)\} = \Lambda\left\{\det\left(\begin{bmatrix} s\Sigma - (A_{11} - A_{12}A_{22}^{-1}A_{21}) & \mathbf{0} \\ -A_{21} & -A_{22}\end{bmatrix}\right)\right\}. \tag{30}$$

Further, pre-multiplying $\begin{bmatrix} \Sigma^{-1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$ with the matrices in (30), and using the definition of A_d in (13), following holds:

$$\Lambda\{\det(sE-A)\} = \Lambda\left\{\det\left(\begin{bmatrix} sI - A_d & \mathbf{0} \\ -A_{21} & -A_{22}\end{bmatrix}\right)\right\}.$$

Since A_{22} is nonsingular (see the discussion in proof of Theorem 3.1),

$$\Lambda\{\det(sE-A)\} = \Lambda\{\det(sI-A_d)\}.$$

This completes the proof. □

Definition of matrix L:

$$
L := \left[\begin{array}{cccc|ccc}
\sum_{j=2}^{n+m} B_{1j} & -B_{12} & \cdots & -B_{1n} & -B_{1(n+1)} & \cdots & -B_{1(n+m)} \\
-B_{21} & \sum_{\substack{j=1\\ j\neq 2}}^{n+m} B_{2j} & \cdots & -B_{2n} & -B_{2(n+1)} & \cdots & -B_{2(n+m)} \\
\vdots & \vdots & \ddots & \vdots & & \vdots & \\
-B_{n1} & -B_{n2} & \cdots & \sum_{\substack{j=1\\ j\neq n}}^{n+m} B_{nj} & -B_{n(n+1)} & \cdots & -B_{n(n+m)} \\
\hline
-B_{(n+1)1} & -B_{(n+1)2} & \cdots & -B_{(n+1)n} & \sum_{\substack{j=1\\ j\neq n+1}}^{n+m} B_{(n+1)j} & \cdots & -B_{(n+1)(n+m)} \\
\vdots & \vdots & \ddots & \vdots & & \ddots & \\
-B_{(n+m)1} & -B_{(n+m)2} & \cdots & -B_{(n+m)n} & -B_{(n+m)(n+1)} & \cdots & \sum_{j=1}^{n+m-1} B_{(n+m)j}
\end{array}\right] \tag{31}
$$

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