Parametric Uncertain LTI Systems: An LMI Optimisation Formulation for Robust Feedback Control Design

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ABSTRACT
This article considers the following two problems for a parametric uncertain linear time-invariant (LTI) system. The first problem is as follows: Let a feedback gain matrix be designed such that the closed loop system response satisfies the specified transient performance bounds. Then, compute a ball in the uncertain parameter space such that for all parameter perturbations within it, the closed loop response continue to satisfy the same transient performance bounds. As a solution to this problem, an explicit expression for the radius of ball is provided. The second problem is on the synthesis of a robust static state feedback control, which: i) minimizes the Frobenius norm of gain matrix, ii) maximizes the radius of ball in the uncertain parameter space and iii) ensures achieving specified transient behaviour in the closed loop. For this, a sub-optimal linear matrix inequality (LMI) optimization is formulated by linearizing the associated non-linear matrix inequalities. The desired transient behaviour is achieved by assigning the closed loop poles within some *pre-defined* LMI regions in the complex plane. The efficacy of the developed results are demonstrated with a practical example, where it is observed that the gain matrix, designed via LMI optimization, provides robust transient performance in the closed loop against a wide range of parameter perturbations. In addition, it is highlighted how the proposed methodology can be applied to design a static state feedback control for robust pole clustering of a polytopic uncertain system.

KEYWORDS
Linear time-invariant systems, state feedback, parametric robust control, LMIs.

1. Introduction

The parametric perturbations in a system model occur due to the variations in physical parameters, possibly with change in operating conditions and/or aging. For instance, the load and generation variations cause parametric perturbations in a power network model (Hiskens & Alseddiqui, 2006). Similarly, in an aircraft model, the perturbation occurs with change in mach number, altitude and loading conditions (Ackermann, 1993; Bhattacharyya, Chapellat, & Keel, 1995). In such situations it is natural to investigate whether a controller, designed for the nominal system, performs satisfactorily when there is perturbation in the physical parameters. Another relevant question is the following: Can a feedback control be designed by means of optimization such that the closed loop performance is robust against a wide range of parameter variations. In this work, we address the above questions for a linear time-invariant (LTI) system where we assume that the performance specifications are mentioned in terms of time domain characteristics, such as decay rates (settling time) and/or overshoot (damping

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ratio).

In a static state feedback control, the specified time domain characteristics are achieved by designing a suitable gain matrix such that the closed loop poles are placed at some appropriate locations in the complex plane. Broadly, following two pole placement paradigms are considered in the literature: i) fixed or point-wise and ii) regional. In fixed pole placement approach, the gain matrix is designed to place the closed loop poles at some fixed locations in the complex plane, which are chosen *a priori*. Such algorithms exploit the underlying design freedom in: i) improving the condition number of associated eigenvector matrix so that the poles are robustly placed at specified locations (Kautsky, Nichols, & Dooren, 1985; Mehrmann & Xu, 1997; Rami, Faiz, Benzaouia, & Tadeo, 2009), and ii) reducing the magnitude of control signal (Keel, Fleming, & Bhattacharyya, 1985; Kouvaritakis & Cameron, 1980; Varga, 2000). On the other hand, regional pole placement approach allows the closed loop poles to freely choose their natural positions within a *pre-defined* stability region in the complex plane. The freedom associated with the choice of closed loop poles within a region provides extra flexibility on optimizing several control objectives, such as norm minimization (Datta & Chakraborty, 2014), controller order reduction (Datta, Chakraborty, & Belur, 2016), and H_2/H_∞ performance improvement (Chilali & Gahinet, 1996; Scherer, Gahinet, & Chilali, 1997).

With the above observations, in this work, we consider regional pole placement paradigm for a parametric uncertain LTI system (affine representation of system matrices with respect to uncertain parameters), where the underlying design flexibilities are exploited in: i) minimizing the Frobenious norm of state feedback gain matrix and ii) maximizing the perturbation bounds on the uncertain parameters. The first objective helps in reducing the magnitude of control signal/control effort. Such design philosophy is cost-effective, particularly, where the plants are controlled with expensive actuators (Datta, Chakraborty, & Chaudhuri, 2012). In fact, the effect of measurement noise can also be reduced by using the minimum norm gain matrix (Ackermann, 1993). The second objective provides robust transient performance in the closed loop for a wide range of parameter variations in the system model. The contributions of this work to address the underlying problems are as follows. We first compute a ball in the uncertain parameter space such that the gain matrix, designed for the nominal system, provides satisfactory transient performance for all possible parameter variations within the computed ball. For this, we provide an explicit expression for the radius of ball in the uncertain parameter space. The radius depends on the feedback gain matrix and some positive definite matrices associated with the Lyapunov equations. Acknowledging the fact that the resulting ball is not maximal in the parameter space, it is enlarged through optimization by exploiting the freedom available with the choice of gain matrix and the closed loop poles within a stability region in the complex plane. For this, we propose a novel linearization approach (using Weyl's result and Young's inequality) to obtain relaxed LMIs for the associated non-linear constraints, and then, using a result on LMI regions in the complex plane (Chilali & Gahinet, 1996), the original non-convex problem is formulated as a sub-optimal LMI optimization. Moreover, we show that the proposed methodology can be applied to design a static state feedback control for a polytopic uncertain system for robust pole clustering within a pre-defined stability region.

In the existing literature (see Bhattacharyya et al. (1995); Petersen and Tempo (2014) and the references therein), the parametric uncertain LTI systems are either represented by: i) transfer functions, where the coefficients of the associated polynomials are uncertain, or ii) state-space, where the entries of the system matrices are uncertain. A variety of mathematical methods/algorithms are proposed in the literature for robust stability analysis and synthesis of robust control system for parametric uncertain LTI systems (in transfer function as well as state-space framework). In the state-space framework (where the system matrices depend affinely on uncertain parameters), the bounds on uncertain parameters are computed in Bhattacharyya et al. (1995); Yedavalli (1985); Zhou and Khargonekar (1987), in the form of hypersphere or hypercube, for robust stability analysis. For this, a *common* quadratic Lyapunov function (QLF) approach is used. Further, to enlarge the radius of hypersphere in the uncertain parameter space (referred to it as *robustification*), a non-convex optimization is proposed in Bhattacharyya et al. (1995) in designing a robust output feedback control. Observing that the common QLF approach is conservative in nature, the concept of parameter dependent Lyapunov function (PDLF) is introduced in Feron, Apkarian, and Gahinet (1996); Gahinet, Apkarian, and Chilali (1996) for robust stability analysis and feedback control design. In addition, a probabilistic framework for robust stability analysis is proposed in Calafiore and Campi (2006), which however, has some limitations in robust control synthesis. It is worth mentioning here that the above stated methods only ensure robust stability against parameter perturbations. Considering the fact that *robust transient performance* is more relevant in many practical applications, authors in Chilali, Gahinet, and Apkarian (1999), Bachelier, Arzelier, and Peaucelle (2001) have proposed methodologies for robust transient performance, where one needs to test whether the poles of an uncertain system continue to remain within the given stability region for parameter perturbations (robust Ω-stability analysis). In particular, assuming that the uncertain parameters vary within a unit hypercube, a multiconvex (convex with respect to each of its variables separately) matrix feasibility test is proposed in Chilali et al. (1999), using an affine PDLF, for robust transient performance analysis. Similarly, using PDLF approach, an LMI optimization is formulated in Bachelier et al. (2001) to enlarge the bounds on uncertain parameters with an assumption that the nominal system satisfies the specified transient performance. This formulation becomes non-linear, leading to a non-convex optimization, if it is of interest to compute a robust static state feedback control.

Another representation of a parametric uncertain system in the state space framework is *polytopic uncertain system*, where the system matrices belong to some polytopes, characterized by the specified vertex matrices. For such systems, robust stability analysis and controller synthesis are performed using QLF approach (Geromel, Peres, & Bernussou, 1991), bounded real lemma framework (He, Wu, & She, 2005; Shaked, 2001) and different variants of PDLF, such as, linear in parameters (Leite & Peres, 2003; M. C. D. Oliveira, Bernussou, & Geromel, 1999; Ramos & Peres, 2002; Rodrigues, Oliveira, & Camino, 2018), quadratic in state and parameters (Bliman, 2004), quadratic in state and linear in parameters (Geromel & Korogui, 2006) and polynomial in parameters (Chesi, Garulli, Tesi, & Vicino, 2005; R. C. L. F. Oliveira & Peres, 2007). The computational complexity of different algorithms for robust stability analysis are studied in Vlassis and Jungers (2014). In addition, the conditions for robust transient performance and robust feedback control design approach are proposed in Arzelier, Henrion, and Peaucelle (2002); Leite and Peres (2003). The existing methodologies for robust stability and robust transient performance analysis are primarily based on finding the existence of a QLF or PDQF, for which, LMI optimizations are formulated. The proposed conditions, however, have some limitations on robust control synthesis, due to the following reasons: i) either the underlying LMI conditions become non-linear (Arzelier et al., 2002; Chesi et al., 2005; Leite & Peres, 2003) or ii) a significantly large number of LMI problems need to be solved (line search over a scalar variable (Dong & Yang, 2013), which is improved in Rodrigues et al. (2018)). In contrast to the existing methodologies, in this work, we propose a novel mathematical framework to design a robust static state feedback control for a parametric uncertain LTI system (affine as well as polytopic) such that the following objectives are achieved: i) the bounds for the uncertain parameters are maximized, ii) the norm of feedback gain matrix is minimized, and iii) robust transient performance against parameter perturbations is guaranteed. Moreover, the underlying problem is formulated as a relaxed LMI optimization, which is numerically tractable and can be solved efficiently with existing

LMI solvers.

The remaining part of the paper is organized as follows. In Section 2, the problems are formulated following to some mathematical preliminaries. A ball in the uncertain parameter space is computed in Section 3. Using this result, an LMI optimization is formulated to minimize the Frobenious norm of gain matrix and maximize the radius of ball in the uncertain parameter space. Moreover, a controller synthesis procedure is proposed for a polytopic uncertain system. The concluding remarks are presented in Section 5, following to some comparative examples and a practical example in Section 4.

Notations: The notation I_n is used to denote an identity matrix of size $n \times n$. The singular values, largest and smallest singular values of A are represented as $\sigma_i(A)$, $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$, respectively. For a square matrix $A \in \mathbb{R}^{n\times n}$, the notation $\lambda_k(A)$, for $k = 1, 2, \cdots, n$, is used for k^{th} eigenvalue of A. For symmetric A, $\lambda_{max}(A)$ and $\lambda_{min}(A)$ denote the maximum and minimum eigenvalues. The notation $A \succ 0$ ($A \succeq 0$, $A \prec 0$) denotes that A is a symmetric positive definite (positive semi-definite, negative definite) matrix. The column-stacking operator of A is denoted as vec(A) whereas $\otimes$ stands for Kronecker Product (Horn & Johnson, 1991). The Frobenius norm and 2-norm of $A \in \mathbb{R}^{m\times n}$ are denoted as $\|A\|_F$ and $\|A\|_2$, respectively. For a scalar α, the notation $|\alpha|$ refers to the absolute value of α. The notation: blkdiag $\{.\}$ refers to a block diagonal matrix.

2. Preliminaries and Problem Formulation

Consider an uncertain linear system whose dynamic behaviour is represented by the following differential equation:

$$\dot{x} = A(\delta)x + B(\delta)u, \tag{1}$$

where $x : \mathbb{R} \to \mathbb{R}^n$ is the state vector, $u : \mathbb{R} \to \mathbb{R}^m$ is input to the system and δ is a real (time-invariant) uncertain parameter. The state matrices $A(\delta)$ and $B(\delta)$ depend affinely on the uncertain parameter δ, and are represented as follows:

$$A(\delta) := A_0 + \delta_1 A_1 + \delta_2 A_2 + \cdots + \delta_r A_r, \tag{2a}$$

$$B(\delta) := B_0 + \delta_1 B_1 + \delta_2 B_2 + \cdots + \delta_r B_r, \tag{2b}$$

where $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m}$, for $i = 1, 2, \cdots, r$, are known fixed matrices which represent the structures of the uncertainty. If $\delta_i = 0$, that is, there is no perturbation in the plant parameters, then (1) is represented as:

$$\dot{x} = A_0 x + B_0 u, \tag{3}$$

where $A_0 \in \mathbb{R}^{n\times n}$ and $B_0 \in \mathbb{R}^{n\times m}$, and referred to it as *nominal system*. With a linear static state feedback control: $u = Fx$, where $F \in \mathbb{R}^{m\times n}$, the nominal system (3) becomes

$$\dot{x} = (A_0 + B_0 F)x, \tag{4}$$

and system (1), with $A(\delta)$ and $B(\delta)$ as in (2), becomes

$$\dot{x} = \left(A_F + \sum_{i=1}^{r} \delta_i (A_i + B_i F) \right) x, \tag{5}$$

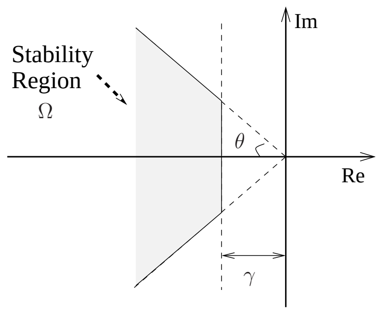

Figure 1. Stability region Ω in the complex plane.

where $A_F = A_0 + B_0F$. Assume that the nominal system (3) is completely controllable, that is, rank $\begin{bmatrix} B_0 \; A_0B_0 \; \cdots \; A_0^{n-1}B_0 \end{bmatrix} = n$. Then, it follows from linear system theory (Antsaklis & Michel, 2006; Wonham, 1974) that the poles of closed loop system (4) can be assigned at any arbitrary locations in the complex plane by designing the gain matrix F appropriately. In this work, we assume that F is designed to place the poles of (4) within a region Ω, as shown in Figure 1, in the complex plane, and is defined as follows (Chilali & Gahinet, 1996):

$$\Omega := \left\{ s \in \mathbb{C} \; : \; \begin{bmatrix} s+\bar{s}+2\gamma & 0 & 0 \\ 0 & \sin\theta(s+\bar{s}) & \cos\theta(s-\bar{s}) \\ 0 & \cos\theta(\bar{s}-s) & \sin\theta(s+\bar{s}) \end{bmatrix} \prec 0 \right\}, \tag{6}$$

where $\mathbb{C}$ denotes the set of all complex numbers and $\bar{s}$ is the complex conjugate of s. Then, the modes of the state response of closed loop system (4) will satisfy the following time domain characteristics: i) minimum decay rate of $\gamma > 0$ (quantifies the settling time t_s) and ii) minimum damping ratio $\zeta = \cos\theta$ where $\theta \in [0, \frac{\pi}{2})$ (quantifies the maximum overshoot).

Note that if the same feedback gain matrix F (designed for (4)) is used in (5), then the poles of (5) may not confine within Ω for any arbitrary values of the uncertain parameters δ_i's. As a result, the transient performance of (5) may deviate from the intended specifications (in fact the system may loose stability). Hence, one of the objectives of this work is to compute the bounds for uncertain parameters δ_i's such that whenever δ_i's vary within the computed bounds, the poles of (5) will remain confined within Ω. For this, we define a parameter vector δ and an open ball $\mathcal{B}(\rho)$ of radius ρ in the parameter space as follows:

$$\delta := \begin{bmatrix} \delta_1 & \delta_2 & \cdots & \delta_r \end{bmatrix}^T \in \mathbb{R}^r, \;\; \mathcal{B}(\rho) := \{\delta \in \mathbb{R}^r : \|\delta\|_2 < \rho\},$$

where $\| \bullet \|_2$ denotes 2-norm of a vector (Watkins, 2002). Define a set: $\mathcal{N} := \{1, 2, \cdots, n\}$. Then, the intended problem is posed as follows.

Problem 1. *Let F be designed such that $\lambda_k(A_F) \in \Omega$, for all $k \in \mathcal{N}$ (one may use* place *command available in MATLAB to compute F by selecting a set of complex numbers, with their conjugates, from Ω). Then, compute the radius ρ of the ball $\mathcal{B}(\rho)$ such that*

$$\lambda_k \left(A_F + \sum_{i=1}^{r} \delta_i (A_i + B_i F) \right) \in \Omega, \;\; \forall\, k \in \mathcal{N} \;\; and \, \forall\, \delta \in \mathcal{B}(\rho). \tag{7}$$

It is shown in the next section that for a specific choice of gain matrix F, one can compute the corresponding ρ such that (7) holds. Recall that the choice of gain matrix F is not unique because of the freedom available in regional pole assignment. Hence, this design flexibility is utilized in: i) maximizing the radius ρ of the ball $\mathcal{B}(\rho)$ in the parameter space, and ii) minimizing the Frobenious norm of F. This leads to the following problem statement.

Problem 2. *Compute the feedback gain matrix F such that: i) $\|F\|_F$ is minimized, ii) the radius ρ of the ball $\mathcal{B}(\rho)$ is maximized, and iii) the relation* (7) *holds for some given γ and θ.*

We will address the above problems in the following section.

3. Main Results

In this section we address Problem 1 and Problem 2, for which, the following quantities are defined for $i = 1, 2, \cdots, r$, and are used in the remaining parts of this article:

$$W_i := A_i X + X A_i^T + B_i F X + X F^T B_i^T,\ \mu := \begin{bmatrix} \|W_1\|_2 & \|W_2\|_2 & \cdots & \|W_r\|_2 \end{bmatrix}^T, \tag{8a}$$

$$\widetilde{A}_i := \begin{bmatrix} A_i \sin\theta & A_i \cos\theta \\ -A_i \cos\theta & A_i \sin\theta \end{bmatrix},\ X_d := \begin{bmatrix} X & \mathbf{0} \\ \mathbf{0} & X \end{bmatrix},\ \widetilde{B}_i := \begin{bmatrix} B_i \sin\theta & B_i \cos\theta \\ -B_i \cos\theta & B_i \sin\theta \end{bmatrix}, \tag{8b}$$

$$F_d := \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & F \end{bmatrix},\quad Z_i := \widetilde{A}_i X_d + X_d \widetilde{A}_i^T + \widetilde{B}_i F_d X_d + X_d F_d^T \widetilde{B}_i^T \tag{8c}$$

$$\xi := \begin{bmatrix} \|Z_1\|_2 & \|Z_2\|_2 & \cdots & \|Z_r\|_2 \end{bmatrix}^T,\ \bar{\delta} := \begin{bmatrix} |\delta_1| & |\delta_2| & \cdots & |\delta_r| \end{bmatrix}^T. \tag{8d}$$

3.1. Computation of ball $\mathcal{B}(\rho)$ in the uncertain parameter space

Consider the closed loop system (4). Let $\lambda_k(A_F) \in \Omega$, for all $k \in \mathcal{N}$. It then follows from Chilali and Gahinet (1996) that there exist symmetric positive definite (SPD) matrices $X \in \mathbb{R}^{n\times n}$, $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{2n\times 2n}$ such that following matrix equations hold:

$$A_F X + X A_F^T + 2\gamma X + P = 0, \tag{9a}$$

$$\begin{bmatrix} (A_F X + X A_F^T)\sin\theta & (X A_F^T - A_F X)^T \cos\theta \\ (X A_F^T - A_F X)\cos\theta & (A_F X + X A_F^T)\sin\theta \end{bmatrix} + Q = 0 \tag{9b}$$

We use relations (9) for computing the radius ρ of $\mathcal{B}(\rho)$ in the next result.

Theorem 1. For a given set of γ and θ for the stability region Ω, let the feedback gain matrix F be designed such that the SPD matrices X, P and Q satisfy (9). Let μ and ξ be as in (8). Then, the radius ρ of the ball $\mathcal{B}(\rho)$, for which (7) holds, is:

$$\rho = \min\left(\frac{\lambda_{\min}(P)}{\|\mu\|_2}, \frac{\lambda_{\min}(Q)}{\|\xi\|_2}\right), \tag{10}$$

where $\min$ stands for minimum.

Proof. Assume that the SPD matrix X, for which (9) holds, satisfies the following matrix inequalities for some fixed values of δ_i's:

$$A_F X + X A_F^T + 2\gamma X + \sum_{i=1}^{r} \delta_i W_i \prec 0 \tag{11a}$$

$$\begin{bmatrix} (A_F X + X A_F^T)\sin\theta & (X A_F^T - A_F X)^T \cos\theta \\ (X A_F^T - A_F X)\cos\theta & (A_F X + X A_F^T)\sin\theta \end{bmatrix} + \sum_{i=1}^{r} \delta_i Z_i \prec 0, \tag{11b}$$

where W_i and Z_i are as in (8). Then, according to Chilali and Gahinet (1996), the eigenvalues $\lambda_k\left(A_F + \sum_{i=1}^{r} \delta_i (A_i + B_i F)\right) \in \Omega$ for all $k \in \mathcal{N}$. Note that the SPD matrices X, P and Q satisfy (9). Hence, the inequalities in (11a) and (11b) can be replaced with: $-P + \sum_{i=1}^{r} \delta_i W_i \prec 0$ and $-Q + \sum_{i=1}^{r} \delta_i Z_i \prec 0$, respectively, which are equivalent to:

$$P - \sum_{i=1}^{r} \delta_i W_i \succ 0, \quad Q - \sum_{i=1}^{r} \delta_i Z_i \succ 0. \tag{12}$$

Then, it follows from the matrix theory (Horn & Johnson, 1991) that the inequalities in (12) hold, if the following relations hold:

$$\lambda_{\max}\left(\sum_{i=1}^{r} \delta_i W_i\right) < \lambda_{\min}(P) \quad \text{and} \quad \lambda_{\max}\left(\sum_{i=1}^{r} \delta_i Z_i\right) < \lambda_{\min}(Q). \tag{13}$$

However, since $\sum_{i=1}^{r} \delta_i W_i$ and $\sum_{i=1}^{r} \delta_i Z_i$ are symmetric matrices, we have the following relations: $|\lambda_j\left(\sum_{i=1}^{r} \delta_i W_i\right)| = \sigma_j(\sum_{i=1}^{r} \delta_i W_i)$, for $j = 1, 2, \cdots, n$, and $|\lambda_l(\sum_{i=1}^{r} \delta_i Z_i)| = \sigma_l(\sum_{i=1}^{r} \delta_i Z_i)$, for $l = 1, 2, \cdots, 2n$, and hence (Horn & Johnson, 1991; Watkins, 2002):

$$\max_j \left|\lambda_j\left(\sum_{i=1}^{r} \delta_i W_i\right)\right| = \sigma_{\max}\left(\sum_{i=1}^{r} \delta_i W_i\right) = \left\|\sum_{i=1}^{r} \delta_i W_i\right\|_2, \tag{14a}$$

$$\max_l \left|\lambda_l\left(\sum_{i=1}^{r} \delta_i Z_i\right)\right| = \sigma_{\max}\left(\sum_{i=1}^{r} \delta_i Z_i\right) = \left\|\sum_{i=1}^{r} \delta_i Z_i\right\|_2. \tag{14b}$$

Furthermore, since $\lambda_{\max}\left(\sum_{i=1}^{r} \delta_i W_i\right) \le \max_j |\lambda_j\left(\sum_{i=1}^{r} \delta_i W_i\right)|$ and $\lambda_{\max}\left(\sum_{i=1}^{r} \delta_i Z_i\right) \le \max_l |\lambda_l\left(\sum_{i=1}^{r} \delta_i Z_i\right)|$, it is clear from (13) and (14) that the inequalities in (12) hold, if:

$$\left\|\sum_{i=1}^{r} \delta_i W_i\right\|_2 < \lambda_{\min}(P) \quad \text{and} \quad \left\|\sum_{i=1}^{r} \delta_i Z_i\right\|_2 < \lambda_{\min}(Q). \tag{15}$$

Further, using the definitions of $\bar{\delta}$ and μ as in (8), and the matrix norm properties (Watkins, 2002), we have: $\|\sum_{i=1}^{r} \delta_i W_i\|_2 \le \sum_{i=1}^{r} |\delta_i|\ \|W_i\|_2 = \bar{\delta}^T \mu$. Similarly, following relation can be obtained: $\|\sum_{i=1}^{r} \delta_i Z_i\|_2 \le \sum_{i=1}^{r} |\delta_i|\ \|Z_i\|_2 = \bar{\delta}^T \xi$. Since the elements of $\bar{\delta}$, μ and ξ are positive numbers, $\bar{\delta}^T \mu = |\bar{\delta}^T \mu|$ and $\bar{\delta}^T \xi = |\bar{\delta}^T \xi|$. Moreover, according to the Cauchy-Schwarz inequality (Watkins, 2002): $|\bar{\delta}^T \mu| \le \|\bar{\delta}\|_2 \|\mu\|_2$ and $|\bar{\delta}^T \xi| \le \|\bar{\delta}\|_2 \|\xi\|_2$. Hence, it follows from (15) that the relation $\|\sum_{i=1}^{r} \delta_i W_i\|_2 < \lambda_{\min}(P)$ will be satisfied if the following relation holds: $\|\bar{\delta}\|_2 < \frac{\lambda_{\min}(P)}{\|\mu\|_2}$. Similarly, $\|\sum_{i=1}^{r} \delta_i Z_i\|_2 < \lambda_{\min}(Q)$ will be satisfied if:

$\|\bar{\delta}\|_2 < \frac{\lambda_{\min}(Q)}{\|\xi\|_2}$. It is then clear that if

$$\|\bar{\delta}\|_2 < \min\left(\frac{\lambda_{\min}(P)}{\|\mu\|_2}, \frac{\lambda_{\min}(Q)}{\|\xi\|_2}\right), \tag{16}$$

then both the matrix inequalities in (12) hold, which implies that the SPD matrix X satisfies (11). Hence, the poles of closed loop uncertain system (5) will lie within Ω whenever $\bar{\delta}$ satisfies (16). Since $\|\delta\|_2 = \|\bar{\delta}\|_2$, it conveys that we can set the radius ρ for the ball $\mathcal{B}(\rho)$ as in (10). Hence, the proof is completed. □

Remark 1. The procedure we have followed in the proof of Theorem 1 can be extended in a similar way to compute a ball for a larger class of stability regions Ω that are obtained by the intersection of several LMI regions (see Chilali et al. (1999) for such regions) in the complex plane. This will then allow to accommodate a wider class of transient specifications for system (5).

Remark 2. According to Theorem 1, every feedback gain matrix F, which is designed for assigning the poles of system (4) within a stability region Ω, ensures robustness against parameter perturbations. However, the amount of allowable perturbations depends on the matrices: X, P, Q and F.

Note that the ball $\mathcal{B}(\rho)$ computed in Theorem 1 may not be maximal in the parameter space. Since the choice of gain matrix F is not unique, we compute it through optimization such that $\mathcal{B}(\rho)$ is maximized.

3.2. Optimization formulation for controller synthesis

In this section, we formulate an LMI optimization to address Problem 2. The freedom available with the choice of feedback gain matrix F, and the closed loop poles within a stability region Ω are exploited to minimize $\|F\|_F$ and maximize the radius ρ of $\mathcal{B}(\rho)$. For this, following quantities are defined, for $i = 1, 2, \cdots, r$, which are used in the forthcoming results:

$$\Delta_\gamma := I_n \otimes \delta^*, \;\; \Delta_\theta := I_{2n} \otimes \delta^*, \;\; \phi = [\phi_1 \; \phi_2 \; \cdots \; \phi_{r-1}]^T, \;\; \Phi := I_n \otimes \phi, \tag{17a}$$

$$\psi = [\psi_1 \; \psi_2 \; \cdots \; \psi_{r-1}]^T, \;\; \Psi := I_{2n} \otimes \psi, \;\; Y := FX, \; Y_d := \text{blkdiag}\{Y, Y\}, \tag{17b}$$

$$W_{Y_i} := A_i X + X A_i^T + B_i Y + Y^T B_i^T, \; Z_{Y_i} := \widetilde{A}_i X_d + X_d \widetilde{A}_i^T + \widetilde{B}_i Y_d + Y_d^T \widetilde{B}_i^T, \tag{17c}$$

where $\delta^* = [\delta_1^* \; \delta_2^* \; \cdots \; \delta_r^*]^T$, and δ_i^*, ψ_i, ϕ_i are some real numbers. It is also assumed that the eigenvalues of all the symmetric matrices are arranged in decreasing order.

Lemma 1. *Let W_i and Z_i be as defined in* (8)*. Let Δ_γ, Δ_θ, ϕ, ψ, Φ and Ψ be as defined in* (17)*, where δ_i^*, ψ_i, ϕ_i are some real numbers, and are considered to be variables. Assume* *that the following matrix inequalities hold:*

(i) *blkdiag* $\{\phi_1 I_n - W_1, \phi_2 I_n - W_2, \cdots, \phi_{r-1} I_n - W_{r-1}\} \succeq 0,$

(ii) *blkdiag* $\{\psi_1 I_{2n} - Z_1, \psi_2 I_{2n} - Z_2, \cdots, \psi_{r-1} I_{2n} - Z_{r-1}\} \succeq 0,$

$$(iii) \begin{bmatrix} P & W_r^T & \Delta_\gamma^T & \Phi^T \\ W_r & 2I_n & \mathbf{0} & \mathbf{0} \\ \Delta_\gamma & \mathbf{0} & 2I_{nr} & \mathbf{0} \\ \Phi & \mathbf{0} & \mathbf{0} & 2I_{n(r-1)} \end{bmatrix} \succ 0, \quad (iv) \begin{bmatrix} Q & Z_r^T & \Delta_\theta^T & \Psi^T \\ Z_r & 2I_{2n} & \mathbf{0} & \mathbf{0} \\ \Delta_\theta & \mathbf{0} & 2I_{2nr} & \mathbf{0} \\ \Psi & \mathbf{0} & \mathbf{0} & 2I_{2n(r-1)} \end{bmatrix} \succ 0.$$

Then, $P \succ 0$*,* $P - \sum_{i=1}^{r} \delta_i^* W_i \succ 0$*,* $Q \succ 0$ *and* $Q - \sum_{i=1}^{r} \delta_i^* Z_i \succ 0$*.*

Proof. According to the definition, W_i is a symmetric matrix, and hence, its eigenvalues $\lambda_k(W_i)$ are real. Assume that the eigenvalues of all the symmetric matrices are arranged in decreasing order. Then, applying the Weyl's result (Horn & Johnson, 1987, Theorem 4.3.1) (eigenvalues of addition of the two symmetric matrices), following relations are obtained for each $k \in \mathcal{N}$:

$$\begin{aligned} \lambda_k\left(\sum_{i=1}^{r} \delta_i^* W_i\right) &\leq \delta_1^* \lambda_{\max}(W_1) + \lambda_k\left(\sum_{i=2}^{r} \delta_i^* W_i\right) \\ &\leq \delta_1^* \lambda_{\max}(W_1) + \delta_2^* \lambda_{\max}(W_2) + \lambda_k\left(\sum_{i=3}^{r} \delta_i^* W_i\right) \\ &\vdots \\ &\leq \delta_1^* \lambda_{\max}(W_1) + \delta_2^* \lambda_{\max}(W_2) + \cdots + \delta_{r-1}^* \lambda_{\max}(W_{r-1}) + \delta_r^* \lambda_k(W_r) \\ &\leq |\delta_1^* \lambda_{\max}(W_1)| + |\delta_2^* \lambda_{\max}(W_2)| + \cdots + |\delta_{r-1}^* \lambda_{\max}(W_{r-1})| + |\delta_r^* \lambda_k(W_r)|. \end{aligned} \tag{18}$$

Since the inequality (i) holds, we have: $\lambda_{\max}(W_j) \leq \phi_j$, for $j = 1, 2, \cdots, r-1$. Hence, from (18), following relation can be written:

$$\begin{aligned} \lambda_k\left(\sum_{i=1}^{r} \delta_i^* W_i\right) &\leq |\delta_1^* \phi_1| + |\delta_2^* \phi_2| + \cdots + |\delta_{r-1}^* \phi_{r-1}| + |\delta_r^* \lambda_k(W_r)| \\ &= |\delta_1^*||\phi_1| + |\delta_2^*||\phi_2| + \cdots + |\delta_{r-1}^*||\phi_{r-1}| + |\delta_r^*||\lambda_k(W_r)|. \end{aligned} \tag{19}$$

Since $|\delta_i^*|$, $|\phi_i|$ and $|\lambda_k(W_r)|$ are positive real numbers, following relations are obtained by applying the Young's inequality to the individual summands (right hand side) of (19):

$$\begin{aligned} \lambda_k\left(\sum_{i=1}^{r} \delta_i^* W_i\right) &\leq \frac{|\delta_1^*|^2}{2} + \frac{|\phi_1|^2}{2} + \cdots + \frac{|\delta_{r-1}^*|^2}{2} + \frac{|\phi_{r-1}|^2}{2} + \frac{|\delta_r^*|^2}{2} + \frac{|\lambda_k(W_r)|^2}{2} \\ &= \frac{1}{2}\|\delta^*\|_2^2 + \frac{1}{2}\|\phi\|_2^2 + \frac{1}{2}|\lambda_k(W_r)|^2. \end{aligned} \tag{20}$$

Moreover, since inequality (ii) holds, we have: $\lambda_{\max}(Z_j) \leq \psi_j$, for $j = 1, 2, \cdots, r-1$.

Hence, similar to the above procedure, it can be shown that:

$$\lambda_k \left(\sum_{i=1}^{r} \delta_i^* Z_i \right) \leq \frac{1}{2} \|\delta^*\|_2^2 + \frac{1}{2} \|\psi\|_2^2 + \frac{1}{2} |\lambda_k(Z_r)|^2, \quad \text{for } k \in \mathcal{N}. \tag{21}$$

Further, using the Schur complement relation (Boyd, Ghaoui, Feron, & Balakrishnan, 1994), the inequality (iii) is equivalent to:

$$P \succ 0, \quad \text{and} \quad P - \frac{1}{2} \left(W_r^T W_r + \Delta_\gamma^T \Delta_\gamma + \Phi^T \Phi \right) \succ 0.$$

Then, it follows that $\lambda_k \left(\frac{1}{2} (W_r^T W_r + \Delta_\gamma^T \Delta_\gamma + \Phi^T \Phi) \right) < \lambda_k(P)$ for all $k \in \mathcal{N}$ (Horn & Johnson, 1987, Problem 14, Section 4.3). According to the definition of Δ_γ and Φ, we have: $\Delta_\gamma^T \Delta_\gamma = \|\delta^*\|_2^2 I_n$ and $\Phi^T \Phi = \|\phi\|_2^2 I_n$. Since W_r is symmetric, we have: $\lambda_k(W_r^T W_r) = |\lambda_k(W_r)|^2$. Hence, for $k \in \mathcal{N}$:

$$\lambda_k \left(\frac{1}{2} (W_r^T W_r + \Delta_\gamma^T \Delta_\gamma + \Phi^T \Phi) \right) = \left(\frac{1}{2} (|\lambda_k(W_r)|^2 + \|\delta^*\|_2^2 + \|\phi\|_2^2) \right) < \lambda_k(P). \tag{22}$$

With the similar procedure, since the inequality (iv) holds, it can be shown that

$$\lambda_k \left(\frac{1}{2} (Z_r^T Z_r + \Delta_\theta^T \Delta_\theta + \Psi^T \Psi) \right) = \left(\frac{1}{2} (|\lambda_k(Z_r)|^2 + \|\delta^*\|_2^2 + \|\psi\|_2^2) \right) < \lambda_k(Q). \tag{23}$$

Hence, it follows from (20) and (22) that $\lambda_k \left(\sum_{i=1}^{r} \delta_i^* W_i \right) < \lambda_k(P)$. Further, since (21) and (23) hold, we have: $\lambda_k \left(\sum_{i=1}^{r} \delta_i^* Z_i \right) < \lambda_k(Q)$. This implies, if the matrix inequalities (i) and (iii) hold, then $P \succ 0$ and $P - \sum_{i=1}^{r} \delta_i^* W_i \succ 0$. Similarly, if the inequalities (ii) and (iv) hold, then $Q \succ 0$ and $Q - \sum_{i=1}^{r} \delta_i^* Z_i \succ 0$. This completes the proof. □

Remark 3. In the presence of a single uncertain parameter δ_1 (for $r = 1$) in (2), the conditions of Lemma 1 can be simplified as follows. Since $\lambda_k (\delta_1^* W_1) \leq |\delta_1^*| |\lambda_k (W_1)|$ and $\lambda_k (\delta_1^* Z_1) \leq |\delta_1^*| |\lambda_k (Z_1)|$, by applying Young's inequality, we have: $\lambda_k (\delta_1^* W_1) \leq \frac{1}{2} |\delta^*|_1^2 + \frac{1}{2} |\lambda_k(W_1)|^2$ and $\lambda_k (\delta_1^* Z_1) \leq \frac{1}{2} |\delta^*|_1^2 + \frac{1}{2} |\lambda_k(Z_1)|^2$. It then directly follows from the proof of Lemma 1 that if the following matrix equations hold:

$$(i) \quad \begin{bmatrix} P & W_1^T & \Delta_\gamma^T \\ W_1 & 2I_n & \mathbf{0} \\ \Delta_\gamma & \mathbf{0} & 2I_n \end{bmatrix} \succ 0, \quad (ii) \quad \begin{bmatrix} Q & Z_1^T & \Delta_\theta^T \\ Z_1 & 2I_{2n} & \mathbf{0} \\ \Delta_\theta & \mathbf{0} & 2I_{2n} \end{bmatrix} \succ 0,$$

then $P \succ 0$, $P - \delta_1^* W_1 \succ 0$, $Q \succ 0$ and $Q - \delta_1^* Z_1 \succ 0$ (there is no requirement of matrix inequalities (i) and (ii) of Lemma 1).

Using Lemma 1, we propose the following result, which is used to address Problem 2.

Theorem 2. *For a given uncertain system* (1)*, let $u = Fx$ be the feedback control. Assume that the transient specifications for the system response are specified in terms of γ and ζ (or θ). Assume that the nominal system, that is, the pair (A_0, B_0) is completely controllable. Let the matrices: Y, W_{Y_i}, Z_{Y_i}, Δ_γ, Δ_θ, Φ and Ψ be as defined in* (17)*. Assume that the following* *matrix equalities and inequalities hold for the variable set:* $\mathcal{V} := \{X, P, Q, Y, \phi, \psi, \delta^*, \beta, \alpha\}$*:*

$$(i)\ \textit{blkdiag}\left\{\phi_1 I_n - W_{Y_1}; \phi_2 I_n - W_{Y_2}; \cdots; \phi_{r-1} I_n - W_{Y_{r-1}}\right\} \succeq 0,$$
$$(ii)\ \textit{blkdiag}\left\{\psi_1 I_{2n} - Z_{Y_1}; \psi_2 I_{2n} - Z_{Y_2}; \cdots; \psi_{r-1} I_{2n} - Z_{Y_{r-1}}\right\} \succeq 0,$$
$$(iii) \begin{bmatrix} P & W_{Y_r}^T & \Delta_\gamma^T & \Phi^T \\ W_{Y_r} & 2I_n & \mathbf{0} & \mathbf{0} \\ \Delta_\gamma & \mathbf{0} & 2I_{nr} & \mathbf{0} \\ \Phi & \mathbf{0} & \mathbf{0} & 2I_{n(r-1)} \end{bmatrix} \succ 0, \quad (iv) \begin{bmatrix} Q & Z_{Y_r}^T & \Delta_\theta^T & \Psi^T \\ Z_{Y_r} & 2I_{2n} & \mathbf{0} & \mathbf{0} \\ \Delta_\theta & \mathbf{0} & 2I_{2nr} & \mathbf{0} \\ \Psi & \mathbf{0} & \mathbf{0} & 2I_{2n(r-1)} \end{bmatrix} \succ 0,$$
$$(v)\ M_Y + 2\gamma X + P = 0, \quad (vi) \begin{bmatrix} M_Y \sin\theta & N_Y^T \cos\theta \\ N_Y \cos\theta & M_Y \sin\theta \end{bmatrix} + Q = 0,$$
$$(vii) \begin{bmatrix} \mathbf{X}_D & \beta I_{mn} \\ \beta I_{mn} & I_{mn} \end{bmatrix} \succ 0, \quad (viii) \begin{bmatrix} \alpha I_{mn} & vec(Y^T) \\ vec(Y^T)^T & \alpha \end{bmatrix} \succ 0,$$

where $M_Y = A_0 X + X A_0^T + B_0 Y + Y^T B_0^T$, $N_Y = X A_0^T - A_0 X + Y^T B_0^T - B_0 Y$ *and* $\mathbf{X}_D = \textit{blkdiag}\{X, X, \cdots, X\}$*. Then, the following statements hold:*

(1) $\lambda_k \left(A_F + \sum_{i=1}^r \delta_i (A_i + B_i F)\right) \in \Omega, \ \forall\, k \in \mathcal{N}$ *and* $\forall\, \delta \in \mathcal{B}(\rho)$*, where* $\rho = \|\delta^*\|_2$*,*
(2) the Frobenious norm of gain matrix: $\|F\|_F < \frac{\alpha}{\beta^2}$*.*

Proof. Corresponding to the specified γ and ζ (or θ), let Ω be the desired stability region. Now, replace Y with FX in the above matrix inequalities and equalities. Then, since the matrix inequalities (i), (ii), (iii) and (iv) hold, it follows from Lemma 1 that:

$$P \succ 0, \ P - \sum_{i=1}^r \delta_i^* W_i \succ 0, \ Q \succ 0, \ Q - \sum_{i=1}^r \delta_i^* Z_i \succ 0. \tag{24}$$

Further, since the nominal system is completely controllable, the matrix equations (v) and (vi) will hold. Then, using (24) and matrix equations (v) and (vi), it follows from the proof of Theorem 1 that for a particular $\delta = \delta^*$, the matrix inequalities in (11) hold. This implies, according to Chilali and Gahinet (1996), for every $k \in \mathcal{N}$, $\lambda_k \left(A_F + \sum_{i=1}^r \delta_i^* (A_i + B_i F)\right) \in \Omega$. Further, since the inequalities (iii) and (iv) hold, it follows from the proof of Lemma 1 that:

$$\frac{1}{2}\left(|\lambda_k(W_r)|^2 + \|\delta^*\|_2^2 + \|\phi\|_2^2\right) < \lambda_k(P), \ \ \frac{1}{2}\left(|\lambda_k(Z_r)|^2 + \|\delta^*\|_2^2 + \|\psi\|_2^2\right) < \lambda_k(Q). \tag{25}$$

Hence, for every δ, which satisfies: $\|\delta\|_2 \leq \|\delta^*\|_2$, following relations hold:

$$\frac{1}{2}\left(|\lambda_k(W_r)|^2 + \|\delta\|_2^2 + \|\phi\|_2^2\right) < \lambda_k(P), \ \ \frac{1}{2}\left(|\lambda_k(Z_r)|^2 + \|\delta\|_2^2 + \|\psi\|_2^2\right) < \lambda_k(Q).$$

This implies: $P - \sum_{i=1}^r \delta_i W_i \succ 0$ and $Q - \sum_{i=1}^r \delta_i Z_i \succ 0$. Hence, the relations in (11) will hold, and we have: $\lambda_k \left(A_F + \sum_{i=1}^r \delta_i (A_i + B_i F)\right) \in \Omega$ for all δ satisfying $\|\delta\|_2 \leq \|\delta^*\|_2$. Therefore, we can set the radius ρ to be $\|\delta^*\|_2$ for the ball $\mathcal{B}(\rho)$. This completes the proof for first statement.

Since $Y = FX$, it can easily be shown that $\text{vec}(F^T) = \mathbf{X}_D^{-1}\text{vec}(Y^T)$. Further, notice that $\|F\|_F = \|\text{vec}(F^T)\|_2$. Hence,

$$\|\text{vec}(F^T)\|_2 = \|\mathbf{X}_D^{-1}\text{vec}(Y^T)\|_2 \leq \|\mathbf{X}_D^{-1}\|_2 \|\text{vec}(Y^T)\|_2. \tag{26}$$

Applying the Schur complement relation to inequality (vii), we have: $\mathbf{X}_D - \beta^2 I_{mn} \succ 0$, which implies: $\lambda_{min}(\mathbf{X}_D) > \beta^2$. Further, it follows from (Horn & Johnson, 1991, Corollary 3.1.5) that $\sigma_{\min}(\mathbf{X}_D) \geq \lambda_{min}(\mathbf{X}_D)$, and hence, $\sigma_{\min}(\mathbf{X}_D) > \beta^2$. Further, since $\|\mathbf{X}_D^{-1}\|_2 = \frac{1}{\sigma_{\min}(\mathbf{X}_D)}$, following relation holds: $\|\mathbf{X}_D^{-1}\|_2 < \frac{1}{\beta^2}$. Applying the Schur complement relation to inequality $(viii)$, we have: $\alpha^2 - \|\text{vec}(Y^T)\|_2^2 > 0$, which implies: $\|\text{vec}(Y^T)\|_2 < \alpha$. Since $\|F\|_F = \|\text{vec}(F^T)\|_2$, the second statement also holds, which follows from (26). □

According to Theorem 2, if the matrix inequalities: (i), (ii), (iii), (iv), (vii), $(viii)$ and equalities: (v), (vi) hold, then: i) $\|F\|_F < \frac{\alpha}{\beta^2}$ and ii) $\lambda_k\left(A_F + \sum_{i=1}^{r} \delta_i(A_i + B_i F)\right) \in \Omega$, for all $k \in \mathcal{N}$ and all $\delta \in \mathcal{B}(\|\delta^*\|_2)$. Hence, they can be considered as constraints in formulating an optimization for Problem 2. Notice that the matrix inequalities and equalities in Theorem 2 are linear with respect to the elements of the variable set $\mathcal{V}$. Further, since $\|F\|_F < \frac{\alpha}{\beta^2}$, $\|F\|_F$ can be minimized by minimizing α and maximizing β, simultaneously. In addition, since $\sum_{i=1}^{r} \delta_i^* \leq \|\delta^*\|_2$, radius $\rho = \|\delta^*\|_2$ can be enlarged by maximizing $\sum_{i=1}^{r} \delta_i^*$. Hence, to address Problem 2, we formulate the following LMI optimization.

Problem 3.

$$\max_{\mathcal{V}} \left(\sum_{i=1}^{r} \delta_i^* + \beta - \alpha \right)$$

such that LMIs: (i), (ii), (iii), (iv), (vii), $(viii)$ *and linear matrix equalities:* (v), (vi) *of Theorem 2 hold for a matrix* Y*, SPD matrices:* X*,* P *and* Q*, vectors:* ϕ*,* ψ *and* δ^**, and scalars:* α *and* β*.*

Since Problem 3 is an LMI optimization, it can be solved with existing LMI solvers. Once Problem 3 is solved, the optimal gain matrix, denoted as F^*, can be computed from the following relation: $F^* = YX^{-1}$.

Remark 4. Note that Problem 3 can be used to address a variety of optimization problems (according to the requirements) as follows by modifying the constraint sets and cost function.

(1) *Computation of largest possible* $\mathcal{B}(\rho)$ *for a* Ω*-stable autonomous system*: Assume that a parametric uncertain autonomous system: $\dot{x} = A(\delta)x$ ($A(\delta)$ is as in (2)), with $\lambda_k(A_0) \in \Omega$, is given. Then, it is of interest to compute largest possible ρ such that $\lambda_k\left(A(\delta)\right) \in \Omega$ for all $k \in \mathcal{N}$ and $\delta \in \mathcal{B}(\rho)$. To address this problem, one needs to solve Problem 3 without constraints: (vii) and $(viii)$ with following modifications: i) set $B_k = \mathbf{0}$, for $k = 0, 1, \cdots, r$ and ii) change the cost function to: $\max_{\mathcal{V}} \ \sum_{i=1}^{r} \delta_i^*$, where $\mathcal{V} := \{X, P, Q, \phi, \psi, \delta^*\}$.

(2) *Maximization of (only) radius* ρ *for parametric uncertain system:* In this case, one would like to compute largest possible ρ for the given system (1) with feedback control $u = Fx$ such that $\lambda_k\left(A_F + \sum_{i=1}^{r} \delta_i^*(A_i + B_i F)\right) \in \Omega$, for all $k \in \mathcal{N}$ and $\delta \in \mathcal{B}(\|\delta^*\|_2)$. For this, Problem 3 has to be solved without constraints: (vii) and $(viii)$ with following modification in the cost function: $\max_{\mathcal{V}} \ \sum_{i=1}^{r} \delta_i^*$, where $\mathcal{V} := \{X, P, Q, Y, \phi, \psi, \delta^*\}$.

(3) *Minimization of* $\|F\|_F$ *for nominal system:* In this case, for a given nominal system (3), one would like to compute F by minimizing $\|F\|_F$ such that $\lambda_k(A_F) \in \Omega$ for all $k \in \mathcal{N}$. For this, Problem 3 has to be solved only with constraints: (vii) and $(viii)$, and following modification in the cost function: $\max_{\mathcal{V}} \ \beta - \alpha$, where $\mathcal{V} := \{X, P, Q, Y, \beta, \alpha\}$.

Remark 5. Note that Problem 3 and the optimization problems listed in Remark 4 are formulated such that both the transient specifications: γ and ζ (or θ) are satisfied. If only one of them needs to be satisfied, then the number of constraints in the LMI optimizations can be reduced further. For instance, if decay rate γ is the only specification, then there is no need of imposing constraints: (ii), (iv) and (vi). Similarly, if minimum damping ratio ζ is the only specification, then there is no need of imposing constraints: (i), (iii) and (v). In addition, if the stability region Ω is only the open left half of complex plane, then constraints: (ii), (iv) and (vi) are not required, and constraint (v) needs to be modified by replacing $\gamma = 0$, in the LMI optimizations. In the presence of a single uncertain parameter δ_1 in (2), that is, for $r = 1$, Problem 3 can further be simplified according to Remark 3. The optimization problem needs to be solved without considering LMIs (i) and (ii), with modifications of LMIs (iii) and (iv) of Theorem 2 as follows: $\begin{bmatrix} P & W_{Y_1}^T & \Delta_\gamma^T \\ W_{Y_1} & 2I_n & \mathbf{0} \\ \Delta_\gamma & \mathbf{0} & 2I_n \end{bmatrix} \succ 0$ and $\begin{bmatrix} Q & Z_{Y_1}^T & \Delta_\theta^T \\ Z_{Y_1} & 2I_{2n} & \mathbf{0} \\ \Delta_\theta & \mathbf{0} & 2I_{2n} \end{bmatrix} \succ 0$, respectively.

3.3. Controller Synthesis for Polytopic Uncertain Plants

In this section, we show that the results developed in Section 3.2 can be applied to design a static state feedback control for a polytopic uncertain plant, which is represented as follows:

$$\dot{x} = A(\delta)x + B(\delta)u, \tag{27}$$

where $A(\delta) := \sum_{i=0}^{r} \delta_i A_i$ and $B(\delta) := \sum_{i=0}^{r} \delta_i B_i$, and the uncertain parameters δ_i, for $i = 0, 1, \cdots, r$, belong to the following set: $\Delta_p := \{\delta_i \in \mathbb{R} \mid \delta_i \geq 0 \text{ and } \sum_{i=0}^{r} \delta_i = 1\}$. The matrices A_i and B_i are vertices of the polytope. Considering the feedback control $u = Fx$, the closed loop system, corresponding to system (27), becomes:

$$\dot{x} = (A(\delta)x + B(\delta)F)\, x. \tag{28}$$

By denoting $A_{cl}(\delta) := A(\delta)x + B(\delta)F = \sum_{i=0}^{r} \delta_i (A_i + B_i F)$, we define the following convex polytope:

$$\mathcal{P}_{cl} := \left\{ A_{cl}(\delta) = \sum_{i=0}^{r} \delta_i (A_i + B_i F) \mid \delta_i \in \Delta_p \right\}. \tag{29}$$

Then, we are interested in designing the feedback gain matrix F such that: i) $\|F\|_F$ is minimized and ii) the eigenvalues of every closed loop system matrices $A_{cl}(\delta) \in \mathcal{P}_{cl}$ cluster in the pre-specified stability region Ω. To address this problem, we first define the following matrices, for $i = 1, 2, \cdots, r$:

$$G_{Y_i} := (A_i - A_0)X + X(A_i^T - A_0^T) + (B_i - B_0)Y + Y^T(B_i^T - B_0^T), \tag{30a}$$
$$H_{Y_i} := (\widetilde{A}_i - \widetilde{A}_0)X_d + X_d(\widetilde{A}_i^T - \widetilde{A}_0^T) + (\widetilde{B}_i - \widetilde{B}_0)Y_d + Y_d^T(\widetilde{B}_i^T - \widetilde{B}_0^T), \tag{30b}$$

where $\widetilde{A}_0 := \begin{bmatrix} A_0 \sin\theta & A_0 \cos\theta \\ -A_0 \cos\theta & A_0 \sin\theta \end{bmatrix}$, $\widetilde{B}_0 := \begin{bmatrix} B_0 \sin\theta & B_0 \cos\theta \\ -B_0 \cos\theta & B_0 \sin\theta \end{bmatrix}$, and other parameters are as defined in (8). Define $\delta_s := \sum_{i=1}^{r} \delta_i$, and represent $A_{cl}(\delta)$ as follows:

$$A_{cl}(\delta) = (1 - \delta_s)(A_0 + B_0 F) + \sum_{i=1}^{r} \delta_i (A_i + B_i F). \tag{31}$$

With respect to the representation of $A_{cl}(\delta)$ as in (31), and for a given vector δ^*, define the following convex polytope:

$$\overline{\mathcal{P}}_{cl} := \{A_{cl}(\delta) \mid \delta_i \geq 0 \text{ and } \|\delta\|_2 < \|\delta^*\|_2 \leq 1\}. \tag{32}$$

Then, we propose the following result.

Theorem 3. Let the matrix $A_{cl}(\delta)$ be represented as in (31). Assume that the matrix pair (A_0, B_0) is completely controllable. For given γ and ζ (specifications for stability region), assume that the following matrix inequalities hold:

$$(i)\ \text{blkdiag}\left\{\phi_1 I_n - G_{Y_1}, \phi_2 I_n - G_{Y_2}, \cdots, \phi_{r-1} I_n - G_{Y_{r-1}}\right\} \succeq 0,$$

$$(ii)\ \text{blkdiag}\left\{\psi_1 I_{2n} - H_{Y_1}, \psi_2 I_{2n} - H_{Y_2}, \cdots, \psi_{r-1} I_{2n} - H_{Y_{r-1}}\right\} \succeq 0,$$

$$(iii)\ \begin{bmatrix} P & G_{Y_r}^T & \Delta_\gamma^T & \Phi^T \\ G_{Y_r} & 2I_n & \mathbf{0} & \mathbf{0} \\ \Delta_\gamma & \mathbf{0} & 2I_{nr} & \mathbf{0} \\ \Phi & \mathbf{0} & \mathbf{0} & 2I_{n(r-1)} \end{bmatrix} \succ 0, \quad (iv)\ \begin{bmatrix} Q & H_{Y_r}^T & \Delta_\theta^T & \Psi^T \\ H_{Y_r} & 2I_{2n} & \mathbf{0} & \mathbf{0} \\ \Delta_\theta & \mathbf{0} & 2I_{2nr} & \mathbf{0} \\ \Psi & \mathbf{0} & \mathbf{0} & 2I_{2n(r-1)} \end{bmatrix} \succ 0,$$

$$(v)\ \begin{bmatrix} I_r & \delta^* \\ (\delta^*)^T & 1 \end{bmatrix} \succeq 0.$$

Further, assume that the following equalities: (v), (vi), and inequalities: (vii), $(viii)$ of Theorem 2 hold for the variable set: $\mathcal{V}$. Then, the following statements hold: i) $\lambda_k\left(A_{cl}(\delta)\right) \in \Omega,\ \forall\, k \in \mathcal{N}$ and $\forall\, A_{cl}(\delta) \in \overline{\mathcal{P}}_{cl}$, and ii) the Frobenious norm of gain matrix: $\|F\|_F < \frac{\alpha}{\beta^2}$.

Proof. Since the inequalities (i), (ii), (iii) and (iv) hold, it directly follows from Lemma 1 that $P \succ 0, P - \sum_{i=1}^r \delta_i^* G_{Y_i} \succ 0, Q \succ 0$ and $Q - \sum_{i=1}^r \delta_i^* H_{Y_i} \succ 0$. Moreover, it follows from the proof of Theorem 2 that for any δ satisfying $\|\delta\|_2 \leq \|\delta^*\|_2$, we have: $P - \sum_{i=1}^r \delta_i G_{Y_i} \succ 0$ and $Q - \sum_{i=1}^r \delta_i H_{Y_i} \succ 0$, which can equivalently be written as:

$$-P + \sum_{i=1}^r \delta_i G_{Y_i} \prec 0, \quad -Q + \sum_{i=1}^r \delta_i H_{Y_i} \prec 0. \tag{33}$$

Since the matrix equations: (v) and (vi) of Theorem 2 hold (due to the controllability assumption on (A_0, B_0)), the inequalities in (33) can be replaced with following:

$$M_Y + 2\gamma X + \sum_{i=1}^r \delta_i G_{Y_i} \prec 0, \quad \begin{bmatrix} M_Y \sin\theta & N_Y^T \cos\theta \\ N_Y \cos\theta & M_Y \sin\theta \end{bmatrix} + \sum_{i=1}^r \delta_i H_{Y_i} \prec 0. \tag{34}$$

Using the definitions of matrices G_{Y_i} and H_{Y_i} as in (30), we have: $G_{Y_i} = W_{Y_i} - M_Y$ and $H_{Y_i} = Z_{Y_i} - \begin{bmatrix} M_Y \sin\theta & N_Y^T \cos\theta \\ N_Y \cos\theta & M_Y \sin\theta \end{bmatrix}$, where Z_{Y_i} is as defined in (17), and M_Y, N_Y are as defined in Theorem 2. Hence, the inequalities (34) can be represented as follows:

$$(1 - \delta_s) M_Y + 2\gamma X + \sum_{i=1}^r \delta_i W_{Y_i} \prec 0, \tag{35a}$$

$$(1 - \delta_s) \begin{bmatrix} M_Y \sin\theta & N_Y^T \cos\theta \\ N_Y \cos\theta & M_Y \sin\theta \end{bmatrix} + \sum_{i=1}^r \delta_i Z_i \prec 0, \tag{35b}$$

where $\delta_s := \sum_{i=1}^{r} \delta_i$. Then, using the definition of $A_{cl}(\delta)$ as in (31), it follows from Chilali and Gahinet (1996) that the eigenvalues $\lambda_k\left(A_{cl}(\delta)\right) \in \Omega,\ \forall\, k \in \mathcal{N}$ and for all $A_{cl}(\delta) \in \overline{\mathcal{P}}_{cl}$. The inequality (v) is to ensure that $\|\delta^*\|_2 \leq 1$. Further, since the constraints (vii) and $(viii)$ (of Theorem 2) hold, it follows from the proof of Theorem 2 that $\|F\|_F < \frac{\alpha}{\beta^2}$. □

The inequalities and matrix equations of Theorem 3 ensure that the eigenvalues of $A_{cl}(\delta) \in \overline{\mathcal{P}}_{cl}$ cluster within the stability region Ω for all perturbations δ_i, satisfying $\|\delta\|_2 \leq \|\delta^*\|_2$. Hence, we can formulate an LMI optimization, similar to Problem 3, by putting the inequalities and matrix equations of Theorem 3 as constraints to design a feedback gain matrix F. When $r = 1$ in (31), the optimization problem can be simplified according to Remark 5. According to the definition, it can be noticed that $\overline{\mathcal{P}}_{cl} \subset \mathcal{P}_{cl}$. Hence, the feedback gain matrix F that we will obtain through the formulated optimization ensures robust clustering of the eigenvalues of a subset of polytopic uncertain plants, characterized by $\overline{\mathcal{P}}_{cl}$.

We now demonstrate the effectiveness of developed methodology with some numerical examples.

4. Numerical Examples

Example 1. In this example, we consider a parametric uncertain system (Bachelier et al., 2001; Zhou & Khargonekar, 1987): $\dot{x} = A(\delta)x = A_0x + \delta_1 A_1 x + \delta_2 A_2 x$, where

$$A_0 = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since there are only two uncertain parameters, we have: $\delta = [\delta_1 \ \ \delta_2]^T$. The eigenvalues of the nominal system are: -1.5858, -4.4142 and -3.0000, and hence, it is a stable autonomous system. We assume that the stability region Ω is open left half of the complex plane. Then, we are interested to compute the largest possible ball $\mathcal{B}(\rho)$ such that $\lambda_k\left(A(\delta)\right) \in \Omega$, for all $k \in \mathcal{N}$ and $\delta \in \mathcal{B}(\rho)$. For this, it follows from Remark 4 (Optimization problem - 1) that Problem 3 needs to be solved without constraints: (vii) and $(viii)$ with following modification to the cost function: $\max\ \delta_1 + \delta_2$. Moreover, since Ω is only the open left half of complex plane, it follows from Remark 5 that constraints: (ii), (iv) and (vi) are also not required, and constraint (v) needs to be modified by replacing $\gamma = 0$. Hence, the modified LMI optimization (with variables: SPD matrices X and P, and vectors ϕ and δ^*) is solved in MATLAB (version R2022a) via LMI solver: $SeDuMi$ (Sturm, 1999), and following quantities are obtained:

$$X = \begin{bmatrix} 0.7456 & 0.0104 & -0.1924 \\ 0.0104 & 0.5594 & -0.0433 \\ -0.1924 & -0.0433 & 0.4007 \end{bmatrix},\ P = \begin{bmatrix} 2.5975 & 0.0085 & 0.0022 \\ 0.0085 & 3.3566 & 0.2664 \\ 0.0022 & 0.2664 & 2.7344 \end{bmatrix},$$

$\delta^* = \begin{bmatrix} 1.1487 & 1.1487 \end{bmatrix}^T$. Hence, according to the discussion in Subsection 3.2, the radius of ball $\mathcal{B}(\rho)$ is: $\rho = \|\delta^*\|_2 = 1.6244$, and $A(\delta)$ is stable for all $\delta \in \mathcal{B}(\rho)$.

This example is also considered in Zhou and Khargonekar (1987) and Bachelier et al. (2001) for computing the range of uncertain parameters δ_1 and δ_2 for robust stability. In Zhou and Khargonekar (1987), the radius of ball $\mathcal{B}(\rho)$ is computed as: $\rho = \|\delta^*\|_2 = 1.6516$. Hence, our result is almost equal to the bound obtained in Zhou and Khargonekar (1987). The approach proposed in Zhou and Khargonekar (1987) can be considered as a particular case of our approach, where the SPD matrix P is chosen as $2I_3$ and W_{Y_i} are set to $\frac{1}{2}(A_iX + XA_i^T)$. In fact, by using $P = 2I_3$ and $W_{Y_i} = \frac{1}{2}(A_i X + X A_i^T)$, our optimization produced the following quantities:

$$X = \begin{bmatrix} 0.5735 & 0.0098 & -0.1471 \\ 0.0098 & 0.3333 & -0.0490 \\ -0.1471 & -0.0490 & 0.2990 \end{bmatrix}, \quad \delta^* = \begin{bmatrix} 1.3196 & 1.3196 \end{bmatrix}^T.$$

For the resulting δ^*, we have: $\|\delta^*\|_2 = 1.8661$, which is greater than the bound (1.6516) obtained in Zhou and Khargonekar (1987). Moreover, notice that the result proposed in Zhou and Khargonekar (1987) for computing the bounds on uncertain parameters is applicable only for robust stability assurance, that is, stability region is only the open left half of complex plane. On the other hand, the methodology proposed in our work helps to compute the robustness bounds on uncertain parameters for a variety of stability regions, which can accommodate minimum decay rate and damping ratio on the modes of system response. A similar problem (robustness with respect to a stability region Ω) is also addressed in Bachelier et al. (2001), where an LMI optimization is proposed to maximize $|\delta_i^*|$. For the considered example, in Bachelier et al. (2001), following robust stability bound is obtained $|\delta_i^*| < 1.7499$. Although, our LMI optimization produced a conservative result, that is, $|\delta_i^*| < 1.1487$, it has following advantages over the formulation given in Bachelier et al. (2001). The LMI optimization formulated in Bachelier et al. (2001) is only applicable for the computation of largest possible bound on the uncertain parameters for a parametric uncertain autonomous system, when the nominal system is Ω-stable (eigenvalues are within Ω). If one needs to use the same formulation for designing a robust static state feedback control, then the proposed formulation (in Bachelier et al. (2001)) will no longer be an LMI optimization, that is, the constraints become non-linear, even after introducing a new matrix variable, as it is suggested in our approach. However, our optimization formulation overcomes this major difficulty, and also allows to accommodate the objective of minimizing the norm of feedback gain matrix, which is important in practical applications.

Example 2. In this example, we consider a vertical take-off and landing helicopter, whose dynamic behaviour, corresponding to a specific loading and flight condition with an airspeed of 135 knots, is represented by (1), where $A(\delta)$ and $B(\delta)$ are as follows (Bhattacharyya et al., 1995, Example 12.4):

$$A(\delta) = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681+\delta_1 & -0.7070 & 1.4200+\delta_2 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}, \quad B(\delta) = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446+\delta_3 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}.$$

It is given that with change in airspeed, the parameters δ_i's, $i = 1, \cdots, 3$, in $A(\delta)$ and $B(\delta)$ vary within the following ranges:

$$|\delta_1| \le 0.05, \quad |\delta_2| \le 0.01, \quad |\delta_3| \le 0.04. \tag{36}$$

The objective is to design a static state feedback control such that following transient performance specifications: i) settling time $t_s \le 20$ seconds, and ii) damping ratio $\zeta \ge 0.35$, are met by the modes of closed loop response for all possible variations of δ_i's within the given bounds in (36). Corresponding to the specified transient specifications, following parameters are set: $\gamma = 0.2$ and $\theta = 69.5127^o$ for the stability region Ω in the complex plane.

To design the feedback gain matrix F, the system matrices $A(\delta)$ and $B(\delta)$ are written in the form of (2) where A_0 and B_0 are obtained by setting $\delta_i = 0$, for $i = 1, \cdots, 3$. Further,

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; B_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$A_3 = \mathbf{0}$, $B_1 = B_2 = \mathbf{0}$. Then, by solving Problem 3 and computing $F^* = YX^{-1}$, following gain matrix is obtained:

$$F^* = \begin{bmatrix} -1.6987 & 2.7828 & 1.7050 & 2.4376 \\ -0.1742 & 2.3290 & 0.7471 & -0.2800 \end{bmatrix},$$

with $\|F^*\|_F = 5.0566$. Further, we obtained: $\delta^* = [0.4091 \;\; 0.4091 \;\; 0.4091]^T$ with $\|\delta^*\|_2 = 0.7086$, and hence, the radius of the ball $\mathcal{B}(\rho)$ is: $\rho = 0.7086$. Hence, according to Theorem 2, the eigenvalues $\lambda_k\left(A(\delta) + B(\delta)F^*\right) \in \Omega$, for all $k \in \mathcal{N}$ and all $\delta \in \mathcal{B}(\rho)$. It is also clear from Figure 2 that all the eigenvalues of perturbed systems are clustered within the specified stability region Ω. Moreover, the state response of perturbed plants are depicted in Figure 3, and it is clear that the responses are settled down within 20 seconds. Corresponding to the specified bounds on parameters δ_i's, as in (36), $\max\|\delta\|_2 = 0.0648$, which is strictly less than the computed bound, that is, 0.7086. Hence, it conveys that the designed gain matrix F^* will produce satisfactory closed loop transient performance under variation of the parameters within the specified bounds in (36). When we solve Problem 3 by setting $P = 2I_3$ and $Q = 2I_6$, and modify the matrices W_{Y_i} and Z_{Y_i} similar to the approach proposed in Zhou and Khargonekar (1987), we obtained an infeasible solution. We also solve Problem 3 according to the procedure given in Remark 4 (Optimization problem - 2), to maximize only the radius ρ. The computed parameters are as follows:

$$F^* = \begin{bmatrix} -61.6847 & 77.9805 & 39.4513 & 94.9557 \\ -17.7460 & 71.1548 & 25.0479 & 34.9210 \end{bmatrix}, \quad \delta^* = [0.6160\ 0.6160\ 0.6160]^T,$$

with $\|F^*\|_F = 166.3844$ and $\|\delta^*\|_2 = \rho = 1.0669$. With this bound, it is verified that all the eigenvalues of perturbed systems are contained within Ω. It can be noticed that for the later case (when there is no requirement of minimization of $\|F\|_F$) the radius of uncertainty ball has increased significantly, and the eigenvalues of perturbed systems remain clustered within the given stability region. Hence, there is a trade-off between the maximization of uncertainty bound and the minimization of Frobenious norm of the feedback gain matrix. However, in all cases the system response will meet the specified transient performance, since the eigenvalues of perturbed systems continue to stay within Ω.

Example 3. In this example we consider a polytopic uncertain system (27), which is characterized by the following set of (vertex) matrices (Arzelier et al., 2002, Example 1):

$$A_0 = \begin{bmatrix} 0 & -0.002 \\ 0.998 & 0 \end{bmatrix}, \; B_0 = \begin{bmatrix} 0.998 \\ 0.002 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & -0.002 \\ 0.002 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.998 \\ 0.998 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & -0.998 \\ 0.998 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.002 \\ 0.002 \end{bmatrix}, \; A_3 = \begin{bmatrix} 0 & -0.998 \\ 0.002 & 0 \end{bmatrix}, \; B_3 = \begin{bmatrix} 0.002 \\ 0.998 \end{bmatrix}.$$

Assuming that the closed loop system matrix $A_{cl}(\delta)$ is written in the form (31), the objective is to design a feedback gain matrix F such that the eigenvalues of $A_{cl}(\delta)$ cluster in the left

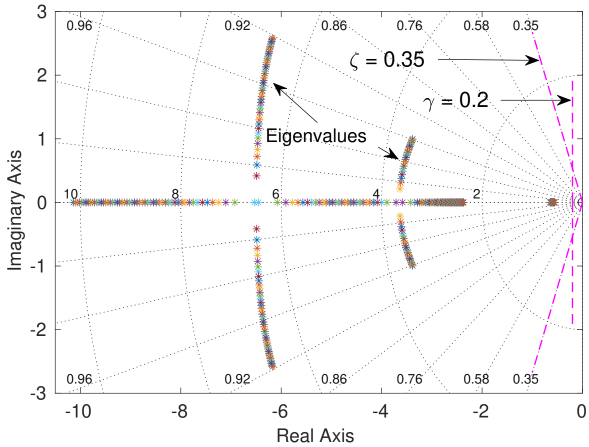

Figure 2. The uncertainty parameters δ_i, for $i = 1, 2, 3$, of Example 2 are allowed to vary within the range $[-0.4091, 0.4091]$, which is equally divided into 100 parts. Then, for each set (out of 100) of δ_i's, the eigenvalues of $(A(\delta)+B(\delta)F^*)$ are computed and plotted here. All the eigenvalues are clustered within the specified stability region.

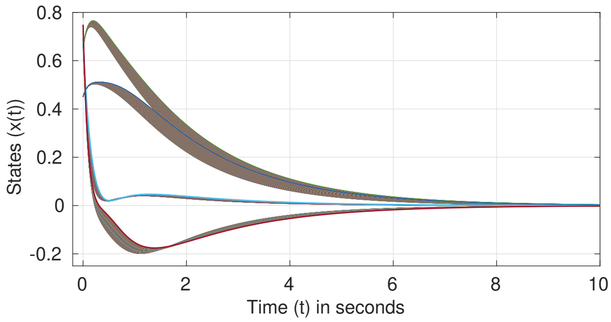

Figure 3. The uncertainty parameters δ_i, for $i = 1, 2, 3$, of Example 2 are allowed to vary within the range $[-0.4091, 0.4091]$, which is equally divided into 100 parts. Then, for each set (out of 100) of δ_i's, the state responses of $\dot{x} = (A(\delta) + B(\delta)F^*)$ are plotted here for a randomly generated initial state.

half of a vertical line at -0.2 ($\gamma = 0.2$) in the complex plane. Since we are not interested in optimizing $\|F\|_F$, and there is no restriction on the transient specification θ, to compute F, we formulate an LMI optimization by setting cost function: $\max \sum_{i=1}^{r} \delta_i^*$ and constraints: inequalities (i), (iii) and (v) of Theorem 3, and equality (v) of Theorem 2. The optimization produced the following quantities: $F^* = [-3.7270 \ -5.2667]$ and $\|\delta^*\|_2 = 0.2549$. To verify the proposed result (Theorem 3), we generate 500 matrices $A_{cl}(\delta)$. For this, we first obtained 500 randomly generated vectors (δ) such that $\|\delta\|_2 \leq \|\delta^*\|_2 = 0.25$. By using the entries of the generated δ's, the matrices $A_{cl}(\delta)$ are constructed as in (31). Note that the generated matrices are elements of the polytope $\overline{\mathcal{P}}_{cl}$ as in (32). Then, the eigenvalues of each $A_{cl}(\delta)$ are plotted, and are depicted in Figure 4. It can be observed that all the eigenvalues are clustered left to the vertical line at -0.2. This example is also considered in Arzelier et al. (2002), where a stabilizing state feedback gain vector (stability region is considered as open left half of the complex plan) is computed. For this, an iterative optimization (to solve a *conic complementarity* problem) is formulated, where the feedback gain vector is obtained as a local solution to the optimization after *four* iterations. In Leite and Peres (2003), an LMI condition is proposed to test if the eigenvalues of the matrices associated with a given polytopic uncertain system belong to a stability region. However, these LMI conditions can not

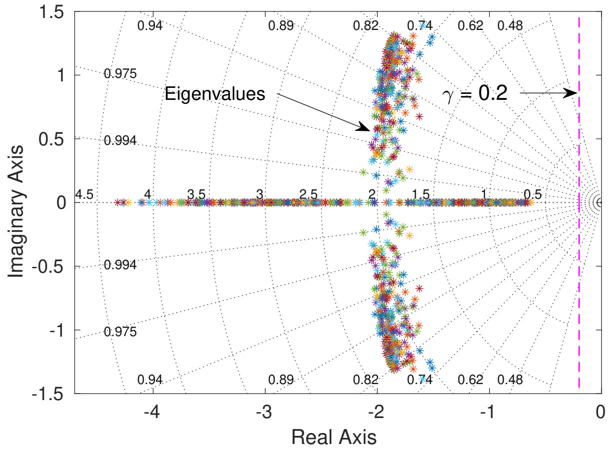

Figure 4. The plot of eigenvalues locations of polytopic uncertain plant matrix $A_{cl}(\delta)$ for Example 3. To construct $A_{cl}(\delta)$ as in (31), the uncertain parameters δ_i's are chosen as entries of 500 randomly generated vectors (δ) such that $\|\delta\|_2 \leq \|\delta^*\|_2 = 0.25$.

directly be applied for feedback control synthesis, since the underlying matrix inequalities become non-linear. The design of an static output feedback control is proposed in Dong and Yang (2013) for stabilizing a polytopic uncertain system. For this, an LMI feasibility problem with *line search over a scalar variable* is proposed, where one needs to solve an LMI feasibility problem repeatedly (often greater than or equal to 100 times) for different values of the scalar variable. One can use this approach to design a stabilizing static state feedback control for a polytopic uncertain plant. However, since a large number of LMI feasibility problems need to be solved to obtain a desired feasible solution, it is computationally inefficient (Sato, 2011). Along the similar direction, an improved result is proposed in Rodrigues et al. (2018) for achieving robust stability, which however, requires line search over two different scalar variables over a bounded search range. On the otherhand, with the methodology proposed in our work, we solve only a single LMI optimization to compute the desired gain vector for the given polytopic uncertain system. Moreover, we could also ensure pole clustering of the polytopic uncertain system within a pre-defined stability region in the complex plane. The limitation of our approach is that the designed gain vector places the eigenvalues of a subset of polytopic uncertain closed loop plants, and one may not be able to place the eigenvalues of all possible polytopic uncertain closed loop plants.

Example 4. In this example we consider a polytopic uncertain system (27), which is characterized by the following two (vertex) matrices (Geromel & Korogui, 2006):

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -12-3d & -12-3d & -25 & -1 \end{bmatrix},\ B_0 = \begin{bmatrix} 0 \\ 0 \\ -6 \\ 6 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -12+3d & -12+3d & -25 & -1 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \end{bmatrix}$$

where d is a free parameter. It is required to design a static state feedback control such that the eigenvalues of $A_{cl}(\delta)$ belong to the open left half of complex plane for each value of $d \geq 0$. For this, we formulate an LMI optimization by setting cost function: $\max \delta_1^*$ with following constraints: i) $X \succ 0$, ii) $\begin{bmatrix} P & G_{Y_1}^T & \Delta_\gamma^T \\ G_{Y_1} & 2I_n & \mathbf{0} \\ \Delta_\gamma & \mathbf{0} & 2I_n \end{bmatrix} \succ 0$, iii) $\begin{bmatrix} 1 & \delta_1^* \\ \delta_1^* & 1 \end{bmatrix} \succeq 0$ and iv) $M_Y + P = 0$ (since there are only two vertex matrices, we have: $r = 1$, and hence, the constraints set has simplified according to Remark 5). The optimization is solved for different values of $d \geq 0$, and we obtained feasible solutions for the following range: $0 \leq d \leq 52$. The feedback

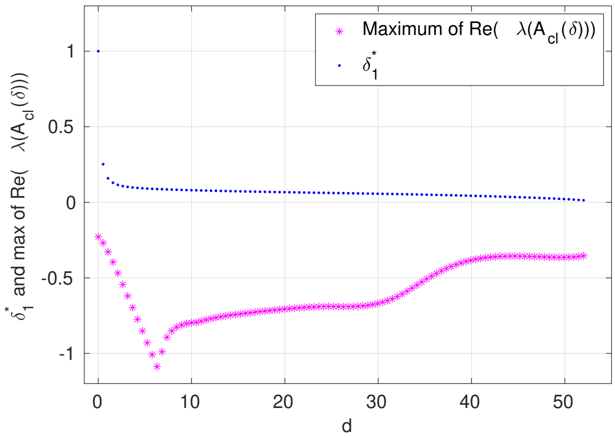

Figure 5. The maximum of real part of the eigenvalues of $A_{cl}(\delta)$ and δ_1^* for Example 4 are represented as $*$ and $\bullet$, respectively. We first obtained evenly spaced 100 points between the range: $0 \le d \le 52$ using *linspace* command in MATLAB. Then, for each d, the eigenvalues of $A_{cl}(\delta)$ is computed by choosing $\delta_1 = \delta_1^* - 0.01$ (just to ensure that $\delta_1 < \delta_1^*$), and the maximum of real part of the eigenvalues of $A_{cl}(\delta)$ are plotted. Further, the computed values of δ_1^* are also plotted.

gain vector for $d = 52$ is computed as follows: $F^* = [31.2747 \;\; 36.0234 \;\; 2.2749 \;\; 0.1067]$ with $\delta_1^* = 0.0133$. For different values of $0 \le d \le 52$, the maximum of real part of the eigenvalues of $A_{cl}(\delta)$ are plotted in Figure 5, which shows that all the eigenvalues of $A_{cl}(\delta)$ are in the open left half of complex plane. Further, the values of δ_1^* with respect to different d is shown in Figure 5, and it is observed that the value of δ_1^* reduces with increase in d. It is highlighted in Geromel and Korogui (2006) that the common QLF approach produces the following range: $0 \le d \le 4$, for which, a stabilizing state feedback control exists for the considered system. Moreover, the methodology proposed in Geromel and Korogui (2006) yields the following range: $0 \le d \le 11.62$, for which, a stabilizing state feedback control exists. Hence, in comparison to these results, our approach produced significantly large range for d. The design approach proposed in Rodrigues et al. (2018) for robust state feedback control can also be used for this example. However, to compute a range for free parameter d for robust stability, one needs to solve LMI optimization with line search (bounded range) over two scalar variables for each d, and thus, leading to a computationally inefficient process. On the otherhand, one needs to solve a single LMI optimization for each choice of d with our approach.

5. Concluding Remarks

In this work, we have considered a parametric uncertain LTI system, and addressed the following problems. First, we compute a ball in the uncertain parameter space such that for all parameter variation within it, the designed gain matrix ensures meeting the specified transient performance in the closed loop. This result also conveys that every feedback gain matrix, designed via a pole placement algorithm, guarantees up to some extent of robustness against parameter variations. As a second problem, we formulate an optimization to compute a gain matrix that minimizes its norm and maximizes the radius of ball in the uncertain parameter space. To get best possible optimum value, regional pole assignment approach (closed loop poles are allowed to take any positions within a stability region in the complex plane) is considered, in contrast to fixed pole assignment. The underlying non-convex optimization is formulated as a relaxed LMI optimization, which is numerically tractable and can be solved efficiently with existing LMI solvers. The proposed results are also extended to design a robust static state feedback control for a polytopic uncertain LTI system. We have obtained satisfactory results through experimentation with numerical examples. It would be interesting to extend the proposed work for designing static output feedback control for time-varying parameter uncertain systems, and addressing the H_2/H_∞ control problems for parameter uncertain systems. We leave these problems for our future research.

The proposed results have potential applications in: i) designing a feedback control, which provides cyber-security to the cyber-physical systems, such as smart power-networks, and ii) constructing cyber-attacks for a cyber-physical system. Nowadays, the smart power-networks are vulnerable to the internet based *load alteration attack* (LAA) (Amini, Pasqualetti, & Mohsenian-Rad, 2018), where an adversary abruptly changes the load parameters of the power-network to cause the power-network unstable or deteriorate its transient performance. To design a feedback control, which can provide cyber-security to the power-network under LAA, one may consider the following approach. Since the load parameters appear in some of the entries of the system matrices in a linearized power-network model, one can consider the power-network model as a parametric uncertain system. Then, using the methodology proposed in this work, a feedback control can be designed for computing a largest ball $\mathcal{B}(\rho)$ in the uncertain parameter space. If an adversary alter the loads that lie within the uncertain parameter ball $\mathcal{B}(\rho)$, then the designed feedback control will ensure meeting the specified transient performance of the power-network under LAA. The proposed results can also be used from the adversary point-of-view, where an adversary is interested in designing a LAA policy for a power-network, similar to the scenario in (Esfahani, Vrakopoulou, Margellos, Lygeros, & Andersson, 2010). By computing the largest possible uncertain ball $\mathcal{B}(\rho)$ in the parameter uncertain space, the adversary can determine the amount of load alteration is required such that the power-network transient performance will get deteriorate or it becomes unstable. These interesting topics need further investigation, which we leave for our future research work.

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