Addendum

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This document is addendum to the following article:

1) S. Datta, "Well-Posedness, Internal Stability and Input-Output Stability in Networked Multi-Agent Systems", *IEEE Control Systems Letters*, Vol.: 6, pp: 1543-1548, 2022.

I. Discussion

In Section IV of the above stated article, the derivation of system matrices for the closed loop system:

$$\dot{x}_{cl} = A_{cl}x_{cl} + B_{cl}r_{cl}, \quad y_{cl} = C_{cl}x_{cl} + D_{cl}r_{cl}. \tag{1}$$

is omitted, which is given here.

II. State space representation of FI-Σ

The closed loop state space realization, from the input r_{cl} to the output y_{cl}, is derived by obtaining relations for e, z, y and u in terms of state variables x and $\bar{x}$, and the external signals: r and w. For this, we use the following Kronecker product properties:

$$(A\otimes B)(C\otimes D) = AC\otimes BD,$$
$$(\alpha A)\otimes B = A\otimes(\alpha B) = \alpha(A\otimes B),$$
$$\alpha A = A\otimes\alpha,$$

where α is a scalar and A, B, C, D are the matrices of compatible sizes.

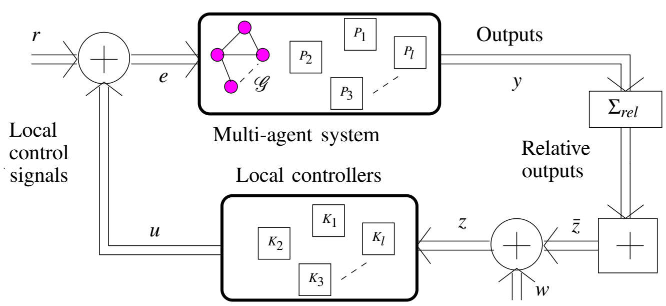

Fig. 1. Feedback interconnection Σ, where r and w are (external) input signals and e, y, z and u are (internal) output signals. The block Σ_{rel} takes the output of neighborhood agents and produces relative outputs.

We have the following set of relations, which are discussed in the published article. By defining a set $\mathscr{N} = \{1, 2, \cdots, l\}$, the input-output relation for each agent is represented by the following relations:

$$y_i(s) = P(s)e_i(s), \quad \forall\ i\in\mathscr{N}, \tag{2}$$

The input-output relation for each controller is represented as follows:

$$u_i(s) = K(s)z_i(s), \quad \forall\ i\in\mathscr{N}. \tag{3}$$

The *neighborhood relative output* information available for i^{th} controller is defined as follows:

$$\bar{z}_i(t) := -\gamma\left[\sum_{j\in N(i)} a_{ij}(y_j(t)-y_i(t))\right] - y_i(t), \tag{4}$$

By defining the vectors: $\bar{z} = [\bar{z}_1\ \bar{z}_2\ \cdots\ \bar{z}_l]^T$ and $y = [y_1\ y_2\ \cdots\ y_l]^T$, (4) can compactly be written as follows:

$$\bar{z} = (\gamma L - I_l)\,y. \tag{5}$$

Note that the input signals for i^{th} agent and its local controller are: $e_i(t)$ and $z_i(t)$, respectively. Let us now define these input signals as follows (for $t \geq 0$):

$$e_i(t) := u_i(t) + r_i(t), \quad z_i(t) := \bar{z}_i(t) + w_i(t), \quad \forall\ i\in\mathscr{N}, \tag{6}$$

where $\bar{z}_i$ is as in (4). The state space representations of agent and its local controller:

$$\dot{x}_i = A_p x_i + b_p e_i, \quad y_i = c_p x_i + d_p e_i, \quad \forall\ i\in\mathscr{N}, \tag{7}$$

and

$$\dot{\bar{x}}_i = A_k\bar{x}_i + b_k z_i, \quad u_i = c_k\bar{x}_i + d_k z_i, \quad \forall\ i\in\mathscr{N}, \tag{8}$$

respectively. Define a matrix $L_g := I_l - \gamma L$, where L is the Laplacian matrix. Then, the following relations are obtained.

- *Relation for e*: According to (6), $z = -L_g y + w$. Hence, by replacing z in the second equation of (8), yields

  $$u = (I_l\otimes c_k)\,\bar{x} + (-L_g\otimes d_k)y + d_k w. \tag{9}$$

  From the second equation of (7),

  $$y = (I_l\otimes c_p)x + (I_l\otimes d_p)e, \tag{10}$$

  and hence, (9) becomes:

  $$u = (I_l\otimes c_k)\,\bar{x} + (-L_g\otimes d_k c_p)x + (-L_g\otimes d_k d_p)e + d_k w. \tag{11}$$

  Since $e = r + u$, from (11), following can be written:

  $$e = G_\infty^{-1}\left[(I_l\otimes c_k)\,\bar{x} + (-L_g\otimes d_k c_p)x + d_k w + r\right]. \tag{12}$$

- *Relation for u*: By replacing $e = r + u$ in (10) yields:

  $$y = (I_l\otimes c_p)x + (I_l\otimes d_p)u + (I_l\otimes d_p)r. \tag{13}$$

  Using (13), (9) becomes:

  $$u = (I_l\otimes c_k)\,\bar{x} + (-L_g\otimes d_k c_p)x$$
  $$+ (-L_g\otimes d_p d_k)u + (-L_g\otimes d_p d_k)r + d_k w,$$

which yields

$$u = G_\infty^{-1}\left[(-L_g \otimes d_k c_p)x + (I_l \otimes c_k)\,\bar{x} + (-L_g d_p d_k)r + d_k w\right]. \tag{14}$$

- *Relation for z*: From the second equation of (8), we can write

$$u = (I_l \otimes c_k)\,\bar{x} + (I_l \otimes d_k)z. \tag{15}$$

Then, by replacing (15) in (13), and using the fact that $z = -L_g y + w$, we have:

$$\begin{aligned} z = -L_g\left[(I_l \otimes c_p)x + (I_l \otimes d_p)\left\{(I_l \otimes c_k)\,\bar{x} + (I_l \otimes d_k)z\right\}\right] \\ + (-L_g d_p)r + w, \end{aligned}$$

which becomes:

$$z = G_\infty^{-1}\left[(-L_g \otimes c_p)x + (-L_g \otimes d_p c_k)\bar{x} + (-L_g d_p)r + w\right]. \tag{16}$$

- *Relation for y*: Using (9), (13) becomes:

$$\begin{aligned} y = (I_l \otimes c_p)x + (I_l \otimes d_p c_k)\bar{x} + (-L_g \otimes d_p d_k)y \\ + (I_l \otimes d_p d_k)w + (I_l \otimes d_p)r, \end{aligned}$$

which can be written as:

$$y = G_\infty^{-1}\left[(I_l \otimes c_p)x + (I_l \otimes d_p c_k)\bar{x} + d_p r + d_p d_k w\right]. \tag{17}$$

From (7) and (8), we have

$$\begin{bmatrix}\dot{x}\\ \dot{\bar{x}}\end{bmatrix} = \begin{bmatrix}I_l \otimes A_p & 0\\ 0 & I_l \otimes A_k\end{bmatrix}\begin{bmatrix}x\\ \bar{x}\end{bmatrix} + \begin{bmatrix}I_l \otimes b_p & 0\\ 0 & I_l \otimes b_k\end{bmatrix}\begin{bmatrix}e\\ z\end{bmatrix}. \tag{18}$$

Define the following matrices: $L_{\bar{d}c} := -L_g \otimes d_k c_p$, $D_{\bar{c}} := I_l \otimes c_k$, $L_c := -L_g \otimes c_p$ and $L_{d\bar{c}} := -L_g \otimes d_p c_k$. Then, using (12) and (16), (18) can be written as:

$$\dot{x}_{cl} = A_{cl} x_{cl} + B_{cl} r_{cl}, \tag{19}$$

where

$$\begin{aligned} A_{cl} = \begin{bmatrix}I_l \otimes A_p & 0\\ 0 & I_l \otimes A_k\end{bmatrix} + \\ \begin{bmatrix}I_l \otimes b_p & 0\\ 0 & I_l \otimes b_k\end{bmatrix}\begin{bmatrix}G_\infty & 0\\ 0 & G_\infty\end{bmatrix}^{-1}\begin{bmatrix}L_{\bar{d}c} & D_{\bar{c}}\\ L_c & L_{d\bar{c}}\end{bmatrix}, \end{aligned} \tag{20a}$$

$$B_{cl} = \begin{bmatrix}I_l \otimes b_p & 0\\ 0 & I_l \otimes b_k\end{bmatrix}\begin{bmatrix}G_\infty & 0\\ 0 & G_\infty\end{bmatrix}^{-1}\begin{bmatrix}I_l & d_k I_l\\ -L_g d_p & I_l\end{bmatrix}. \tag{20b}$$

Finally, by considering (17), (14), (12) and (16) we can write:

$$y_{cl} = C_{cl} x_{cl} + D_{cl} r_{cl}, \tag{21}$$

where

$$C_{cl} = \begin{bmatrix}G_\infty & 0 & 0 & 0\\ 0 & G_\infty & 0 & 0\\ 0 & 0 & G_\infty & 0\\ 0 & 0 & 0 & G_\infty\end{bmatrix}^{-1}\begin{bmatrix}I_l \otimes c_p & I_l \otimes d_p c_k\\ L_{\bar{d}c} & D_{\bar{c}}\\ L_{\bar{d}c} & D_{\bar{c}}\\ L_c & L_{d\bar{c}}\end{bmatrix}, \tag{22a}$$

$$D_{cl} = \begin{bmatrix}G_\infty & 0 & 0 & 0\\ 0 & G_\infty & 0 & 0\\ 0 & 0 & G_\infty & 0\\ 0 & 0 & 0 & G_\infty\end{bmatrix}^{-1}\begin{bmatrix}d_p I_l & d_p d_k I_l\\ -L_g d_p d_k & d_k I_l\\ I_l & d_k I_l\\ -L_g d_p & I_l\end{bmatrix}. \tag{22b}$$