

Leader-Following Synchronization in Singular Multi-Agent Systems Under Signed and Uncertain Network Graph

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Abstract—This article considers the problem of leader-following synchronization in a singular multi-agent system. Two different types of network topologies are considered, one is signed and another is uncertain. A unified reduced-order observer with a distributed feedback control scheme is proposed for both types of network topologies to achieve synchronization. In particular, the following objectives are achieved in the closed-loop: (i) the system is impulse-free, (ii) the estimated states of the agents converge to their respective actual states, and (iii) the follower states synchronize with the leader states, satisfying a *pre-specified decay rate* (helps to regulate the synchronization speed). The developed algorithm provides flexibility to independently design the associated gain parameters. Further, the algorithm utilizes only orthogonal matrices for transformations at various stages, and hence, it is numerically efficient. The effectiveness of the proposed design approach is demonstrated with some examples.

Note to Practitioners—Most of the industry-favorable modeling and simulation software, such as *OpenModelica*, *Simcenter Amesim*, and *MATLAB Simscape*, produce a singular model for a physical system. A network of such singular systems is referred to as *singular multi-agent system* (SMAS). An important control objective of an SMAS is leader-following synchronization, which has potential applications in areas such as vehicle platoons, robotic systems, defense, space, and smart power grids. Such a synchronization objective is achieved through the implementation of a distributed control architecture, where the control action is over a communication network. The communication network, however, is always prone to cyber attacks and uncertainty within communication channels. The attacker may compromise the network and send conflicting signals to make some of the agents behave antagonistically. In this work, we provide a methodology to select control gains and network parameters appropriately so that synchronization can still be achieved despite such network-related issues. Our approach uses a distributed observer-based control strategy that allows practitioners to design the controller when sensor data and communication are limited. Moreover, singular dynamics often exhibits impulsive behavior, which is undesirable in practice. The proposed control architecture also obviates such undesirable behavior and allows regulating the synchronization speed. Since the control gain parameters are designed offline and independently, the proposed control structure is well-suited for real-time implementation in industrial networked systems.

Index Terms—Singular systems, multi-agent systems, observer design, distributed control, leader-follower synchronization.

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I. INTRODUCTION

In recent years, research in the field of multi-agent systems has become popular due to their applications in diverse fields, such as power networks [1], multi-vehicle robotic systems [2], and the space sectors [3]. A multi-agent system consists of a group of dynamical systems, known as agents, and a communication network for inter-agent interaction. The dynamics of an agent is often described as linear state-space systems, transfer functions, non-linear systems, and singular systems (see [4]–[8] and the references therein). In this work, we represent the dynamics of an agent as *singular system*: $E\dot{x} = Ax + Bu$, $y = Cx$, where E is a singular matrix. A singular system allows to include the algebraic equations with differential equations to represent system dynamics. These algebraic equations arise to satisfy certain physical laws and/or constraints. For instance, the states of a power network need to satisfy the *power-flow* algebraic equations [9]. Moreover, *OpenModelica*, an industry-favorable modeling and simulation software, produces a singular state-space representation of a physical system [10]. A multi-agent system having singular agent dynamics is referred to as *singular multi-agent system*.

The dynamic behavior of a singular multi-agent system (SMAS) inherently exhibits the characteristics of a singular system (SS). For instance, the existence of a solution of an SS depends on its *regularity* condition, and the *impulsive behavior* appears under the influence of inconsistent initial conditions and/or non-smoothness of (control) input [11]. The impulsive response is undesirable in practical applications, and hence, it needs to be eliminated through feedback control from an SMAS. Multi-agent systems are often represented through a *graph* for system-level analysis and control design. The vertices of the graph represent the agents, and the edges represent the communication channel between agents [4]. The type of communication among the agents is represented (modeled) by *weights* on the edges. For instance, *cooperative* and *antagonistic* interactions between two agents are modeled as positive and negative edge weights, respectively [12]–[14]. A network graph having both positive and negative edge weights is referred to as *signed graph* [12], [13], [15]. Moreover, often the information shared through the communications network may not be exact, possibly due to the channel capacity constraints, packet drops, communication obstacles, transmission noise, and inexact network model [16], [17]. Such quantities are modeled as channel uncertainties, that is, the edge weights are uncertain parameters that can vary within some bounds. A

network graph with uncertain edge weights is referred to as an *uncertain graph*.

Contributions: In this work, we consider a *single leader* SMAS with two different types of network topologies: one is signed, and the other is uncertain. It is further assumed that all the states of an agent are *not measurable*. With this setup, we address the following problem: design a feedback control for an SMAS, having a signed network topology, such that the follower states synchronize with the leader (leader-following synchronization) without showing impulsive behavior in the closed-loop. Further, the same problem needs to be addressed for an SMAS having an uncertain network topology. The major challenge in these problems is: how to enforce *cooperative behavior* (leader-following synchronization) in an SMAS despite the antagonistic behavior by some of the agents and uncertainty in the communication channel. To address these problems, our major contributions are as follows.

- 1) Since the states of the agents are not measurable, we propose to use a *reduced-order observer* (of rank(E)) for state estimation. For this, a novel projection-based approach is proposed, which utilizes only orthogonal matrices for system transformations and the design process. Hence, the proposed approach is numerically efficient and is scalable for large-order systems.
- 2) A *reduced-order observer-based distributed control* architecture is proposed for an SMAS with a signed network topology. The proposed control scheme ensures that, despite the presence of *antagonistic interaction* among the agents, one can always achieve *cooperative behavior* in the network, that is, leader-following synchronization in the closed-loop SMAS.
- 3) In the proposed design framework, it is shown that one can design the reduced-order observer and controller gain matrices independently, and hence, the standard *separation principle* holds. The other associated gain parameters can also be designed independently by solving a linear matrix inequality (LMI) optimization problem. We further show that it is possible to regulate the *speed of synchronization*, that is, the time required for follower states to converge to leader states in the network.
- 4) The design of an integrated reduced-order observer with distributed control that makes the closed loop SMAS impulse-free is challenging, which we have addressed successfully within our design framework.
- 5) The proposed design framework is also extended for an SMAS with *uncertain network*. For this, the *parametric robust control* framework is used to compute the desired gain parameters and allowable perturbation bounds on edge weights, ensuring that cooperative behavior can still be achieved in the closed-loop.

Applications of the proposed work: Our contributions, particularly in the leader-following synchronization context, have potential applications in areas such as autonomous vehicle platooning, multi-robotic systems, defense, space, and smart power grids. For instance, in a network of small satellites, used for various space applications [3], it is preferred to provide the desired command signals to one of the satellites (leader),

and the others (followers) need to access that command through their nearby satellites, and synchronize with the leader. This helps to minimize the ground station communication. Similarly, in a smart power grid where microgrids are geographically separated, a leader-follower architecture is often preferred, so that the desired command signals (voltage and frequency) can be provided only to the leader microgrid, and the remaining microgrids synchronize with the leader. Moreover, the proposed control scheme is best suited for an SMAS, where cooperative behavior must be preserved in the network despite the antagonistic behavior of some agents. For instance, in a smart power-grid, there is a possibility that a microgrid behaves antagonistically with respect to its neighbors in the following ways: i) by suppressing its generation through local storage, and demanding more power from the other microgrids, ii) by intentionally injecting/absorbing excessive reactive power to the grid, and thereby, putting stress on the other microgrids to maintain appropriate voltage level for their loads, and iii) due to the presence of negative reactance of a transmission line [18], [19]. Furthermore, a networked microgrid system often relies on IoT (Internet-of-Things)-enabled communication networks for coordination, which makes them vulnerable to cyberattacks. In such scenarios, an adversary may intercept communication channels between microgrids and inject malicious data, and/or flip state information to make some microgrids behave antagonistically. Such communication channels are also subjected to a certain level of uncertainty, and hence, the proposed control algorithm is useful in these scenarios as well. In addition, the proposed control architecture emphasizes the use of reduced-order observers, as they are more suitable for large-order systems, such as smart power grids. Lastly, synchronization speed regulation is also an important aspect of our design approach, which is desirable in most networked systems.

Brief literature review and improvements: The existing articles on SMAS primarily consider *consensus* as a cooperative control objective, where relative output-based static feedback control protocols are proposed (see [13], [20], [21], and the references therein). In addition, considering cooperative and antagonistic interaction among the agents, the problem of *bipartite consensus* (states converge to the consensus points: x^* and $-x^*$) is addressed in [13]. Along the same lines, [22], [23] have considered the bipartite *containment consensus* problem for an SMAS, where the network contains multiple leaders. The objective in this framework is to achieve convergence of the follower states to the convex hulls, spanned by the leaders' states. On the other hand, we consider the problem of achieving *cooperative behavior* (impulse-free leader-following synchronization) in an SMAS despite antagonistic agent interactions. We further incorporate how to regulate the synchronization speed in the network. Although these objectives have potential implications in real-world applications, to the best of the authors' knowledge, they are not available in the existing literature on SMAS.

The other control objectives considered for SMAS are consensus with disturbance suppression [24] and output regulation [25]. Most of these algorithms are developed under the assumption that the states of the agents are available for

measurement, which, however, may not always be feasible in practice. In such scenarios, a preferred control architecture is observer-based state feedback control. In [22], [23], full-order observers are used to address bipartite containment consensus in an SMAS. Full-order observers, however, are less preferred for large-scale systems than reduced-order observers (ROOs). The design of ROOs for SSs is proposed in [11], [26]–[29]. Some of these design algorithms compute an $n - p$ order observer [11], [26]–[28], where n is the system order and p is the rank of the output matrix. A numerically efficient algorithm for designing a disturbance decoupling observer is proposed in [30]. These methods, however, do not discuss the design of an integrated controller-observer that satisfies the separation principle. Some other design methods either impose various restrictive assumptions on system matrices [31] or rely on system transformations involving non-singular matrix inversion [28], [29], [32], which may lead to numerical ill-conditioning and are therefore less preferred for large-order systems. On the other hand, our projection-based design approach: i) computes an $\text{rank}(E)$ order observer, which allows the separation principle to hold in the combined controller-observer design, ii) avoids the algebraic variable elimination process, and iii) performs system transformations through orthogonal matrices only. Hence, it is numerically efficient and scalable for large order systems as well. Moreover, robustness against uncertainties in the communication topology of a multi-agent system has been explored in [17], [33] by modeling the edge weights as uncertain parameters. These works, however, are restricted only to single-integrator agent dynamics. In contrast, we consider singular system dynamics, which is a more general system dynamics, and address the leader-follower synchronization problem in an SMAS with an uncertain network.

The paper is organized as follows. Section II presents mathematical preliminaries and problem formulation. Section III presents the design of the reduced-order observer and the observer-based distributed feedback control scheme. Section IV illustrates the approach through numerical examples, including a networked microgrid. Section V concludes the work and outlines future research directions.

Notation: $\otimes$: Kronecker product, $\det(\cdot)$: determinant of a matrix, $\deg\{\cdot\}$: degree of a polynomial, $\text{diag}\{\cdot\}$: a diagonal matrix, I_r : $r \times r$ identity matrix, $\text{eig}(\cdot)$: set of all eigenvalues, $\text{fnp}(\cdot, \cdot)$: set of all finite poles, $\lambda_i(M)$, $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ are i^{th} eigenvalue, maximum eigenvalue and minimum eigenvalue of matrix M , respectively, $\rho(M)$: Spectral radius of M , $\|M\|_2$: 2-norm of matrix M . $M \succ 0$ ($M \prec 0$) denotes a symmetric positive definite (negative definite) matrix.

II. PRELIMINARIES AND PROBLEM FORMULATION

Define the following two sets: $\mathbb{N}_0 := \{0, 1, \dots, r\}$ and $\mathbb{N} := \{1, 2, \dots, r\}$. For $i \in \mathbb{N}_0$, the dynamics of i^{th} agent in an SMAS, consisting of $r + 1$ number of identical agents, is expressed as follows:

$$E\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad y_i(t) = Cx_i(t), \quad t \geq 0, \quad (1)$$

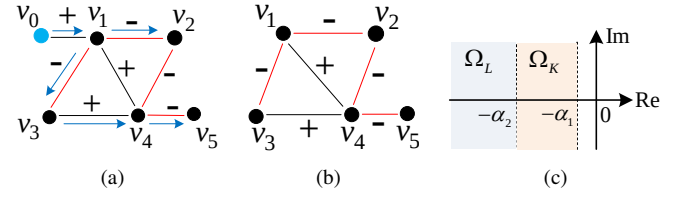


Fig. 1. (a): Signed undirected graph $G_s(\mathcal{V}, \mathcal{E})$ associated with an SMAS with six agents ('+' sign represents cooperative interaction and '-' sign represents antagonistic interaction between two agents). The arrows in (a) represent the flow of leader state x_0 to every follower agent, (b): Subgraph $G_s(\mathcal{V}', \mathcal{E}')$ of graph $G_s(\mathcal{V}, \mathcal{E})$, (c): Stability regions Ω_K (left to the vertical line at $-\alpha_1$) and Ω_L (left to the line at $-\alpha_2$), with $\alpha_2 > \alpha_1 > 0$ in the complex plane.

where the quantities $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$, and $y_i(t) \in \mathbb{R}^p$, represent the state vector, (control) input vector, and output vector of the i^{th} agent, respectively. The following assumptions are considered for each agent (1): i) E is singular and $\text{rank}(E) = q < n$, ii) the pair (E, A) is *regular*, and iii) the triad (E, A, B) is *C-Controllable* and (E, A, C) is *impulse-observable* and *detectable*. The definitions of regularity, observability, and detectability can be found in [11].

In the set of $r + 1$ agents of an SMAS, one agent is specified as the *leader*, while the remaining r agents are considered as *followers*. The dynamics of leader and follower agents are represented by (1) for $i = 0$ and $i \in \mathbb{N}$, respectively. The SMAS is represented by an undirected connected graph $G_s(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{v_0, v_1, \dots, v_r\}$ is the set of vertices, and $\mathcal{E}$ is the set of edges between two vertices. The vertices v_0 and v_i , for $i \in \mathbb{N}$, represent the leader agent and follower agents, respectively. We assign some weight w_{ij} to the edge between two vertices v_i and v_j , for $i, j \in \mathbb{N}$. The weight w_{0i} is assigned to the edge joining vertex v_0 and v_i . Note that we allow the edge weight w_{ij} to be both positive and negative. Hence, $G_s(\mathcal{V}, \mathcal{E})$ is a *signed graph*, as in Figure 1(a).

For a graph $G_s(\mathcal{V}, \mathcal{E})$, a subgraph $G_s(\mathcal{V}', \mathcal{E}')$ is constructed by removing the vertex v_0 and the edges connected to it. The subgraph $G_s(\mathcal{V}', \mathcal{E}')$ only represents the follower agents and the communication among them. We say that follower agent- j is neighbor of the follower agent- i , if there is an edge between v_i and v_j (with edge weight w_{ij}) in $G_s(\mathcal{V}', \mathcal{E}')$. The set of all neighborhood agents j of follower agent- i is denoted as $\mathcal{N}_i$. We define an *adjacency matrix* $\mathcal{A}(G_s) \in \mathbb{R}^{r \times r}$ for the subgraph $G_s(\mathcal{V}', \mathcal{E}')$ by assigning its $(i, j)^{\text{th}}$ element as w_{ij} . Assuming that $w_{ij} = w_{ji}$, and there are no self-loops in the network, the matrix $\mathcal{A}(G_s)$ is symmetric, and its diagonal entries are $w_{ii} = 0$. Since w_{ij} could be positive or negative, the off-diagonal elements of $\mathcal{A}(G_s)$ are combinations of mixed signed non-zero real numbers and zero, which is different from the usual adjacency matrix that represents a cooperative network ($w_{ij} > 0$). We define the *degree matrix* $\mathcal{D}(G_s)$ for $G_s(\mathcal{V}', \mathcal{E}')$ as a diagonal matrix with diagonal elements: $d_i = \sum_{j \in \mathcal{N}_i} w_{ij}$, representing the degree of the vertex v_i . Then, the *Laplacian matrix* $\mathcal{L}$ for $G_s(\mathcal{V}', \mathcal{E}')$ is defined as follows: $\mathcal{L} := \mathcal{D}(G_s) - \mathcal{A}(G_s)$. The Laplacian matrix $\mathcal{L}$ is symmetric, and the elements in a row (or column) sum to zero. However, $\mathcal{L}$ has the following properties:

- the diagonal and off-diagonal elements can be of mixed-signed non-zero real numbers or zero,
- the non-zero eigenvalues of $\mathcal{L}$ can be positive or negative real numbers.

These properties are different from the usual Laplacian matrix defined for a cooperative network ($w_{ij} > 0$). With this setup, we first address the following problem.

Problem 1: Consider an SMAS with agent dynamics as in (1). Let $G_s(\mathcal{V}, \mathcal{E})$ be a signed graph, and it is connected. Let $\hat{x}_i$ be the estimated state of i^{th} agent. Then, design an observer plus control scheme such that: i) the closed-loop system is impulse-free, ii) the estimated state $\hat{x}_i$ converges to actual state $x_i(t)$ of i^{th} agent, for all $i \in \mathbb{N}_0$, and iii) the state trajectories of the follower agents $x_i(t)$, for all $i \in \mathbb{N}$, synchronize with the leader agent state trajectory $x_0(t)$, satisfying a *pre-specified* decay rate α_1 .

As a second problem, we solve Problem 1 by considering that the network graph is *uncertain*, that is, the edge weights w_{ij} of the network graph are subjected to perturbation.

III. MAIN RESULTS

In this section, we address Problem 1, for which, we propose the following control input $u_i(t)$ for an agent:

$$u_i(t) = u_{m_i}(t) + u_{s_i}(t), \quad \forall i \in \mathbb{N}_0. \quad (2)$$

The control signal $u_{m_i}(t)$ in (2) is defined as follows:

$$u_{m_i}(t) = K_m y_i(t), \quad \text{for all } i \in \mathbb{N}_0, \quad (3)$$

where $K_m \in \mathbb{R}^{m \times p}$ is an output feedback gain matrix. Further, for all $i \in \mathbb{N}_0$, the control signals $u_{s_i}(t)$ are defined as follows:

$$u_{s_0}(t) = 0; \quad u_{s_i}(t) = K \hat{\zeta}_i(t), \quad \forall i \in \mathbb{N}, \quad (4)$$

where $K \in \mathbb{R}^{m \times n}$ is a feedback gain matrix, and the signal $\hat{\zeta}_i(t)$ is defined as follows:

$$\hat{\zeta}_i = \sum_{j \in \mathcal{N}_i} \beta(w_{ij}(\hat{x}_i - \hat{x}_j)) + \gamma_i(h_i(\hat{x}_i - \hat{x}_0)). \quad (5)$$

In (5), the quantities β and γ_i are the *network gain parameters*, which need to be designed, and w_{ij} is the edge weight of signed graph $G_s(\mathcal{V}, \mathcal{E})$. The quantity $h_i = w_{0i}$, if the i^{th} agent is a neighbor of the leader agent. Further, since it is assumed that $G_s(\mathcal{V}, \mathcal{E})$ is connected, the leader state/output information can be transferred to a follower agent through the neighborhood agents, as shown in Figure 1(a). Hence, $h_i = w_{ij}$, for $j \in \mathcal{N}_i$, where $\mathcal{N}_i$ is the neighborhood set of i^{th} agent. Note that there could be multiple agents in the neighborhood of the i^{th} follower agent. For this, one may also prefer to first find a shortest path (least weight path) from the leader to the i^{th} agent, using algorithms such as *Floyd-Warshall algorithm* [34], and then set $h_i = w_{ij}$, where j is the neighborhood agent that appears in the shortest path.

The control law in (3) uses only the *local* output information, whereas the control law in (4) needs both *local* as well as *neighborhood* estimated state information. Since $q = \text{rank}(E) < n$, the agents may show impulsive behavior. Hence, we use control law (3) to make the individual agents

impulse-free first. Using (2) and (3) in (1), for $i \in \mathbb{N}_0$, following closed-loop dynamics is obtained:

$$E\dot{x}_i = (A + BK_m C)x_i + Bu_{s_i}, \quad y_i = Cx_i. \quad (6)$$

Now, (6) can be made impulse-free by appropriately designing the gain matrix K_m . For this, we propose to use the algorithm given in [11, Theorem 7.9]. We later show that implementing a distributed controller for synchronization will not destroy the impulse-free property of the overall closed-loop system. Since the control law u_{s_i} in (4) utilizes estimated states $\hat{x}_i$ of the agents, we next design a reduced-order observer (q^{th} order) for the agents to obtain $\hat{x}_i$.

A. Design of reduced order observer

For an impulse-free agent dynamics (6), a full order descriptor state observer for i^{th} agent is [11, Chapter 11]:

$$E\dot{\hat{x}}_i = (A + BK_m C)\hat{x}_i + Bu_{s_i} + L(y_i - C\hat{x}_i), \quad (7)$$

where $L \in \mathbb{R}^{n \times p}$ is the observer gain matrix. With the assumption that each agent is detectable, one can always design the observer gain matrix L such that finite poles of the pair $(E, A_y - LC)$, where $A_y := A + BK_m C$, can be assigned at any desired stability region in the complex plane. Let the gain matrix L be designed such that the finite poles of $(E, A_y - LC)$ are placed in the stability region Ω_L , as shown in Figure 1(c). By defining a matrix: $A_l := A_y - LC = A + BK_m C - LC$, (7) can be represented as:

$$E\dot{\hat{x}}_i = A_l \hat{x}_i + Bu_{s_i} + Ly_i. \quad (8)$$

Note that the full order observer (8) has two inputs u_{s_i} and y_i . Using these two inputs, let a vector be defined as follows: $u_o = [u_{s_i}^T \quad y_i^T]^T$. Then, (8) becomes:

$$E\dot{\hat{x}}_i = A_l \hat{x}_i + [B \quad L] u_o. \quad (9)$$

Since E is singular, there exists an orthogonal matrix Z such that $ZE = [E_1^T \quad 0^T]^T$. One can obtain such Z by using the singular value decomposition (SVD) of E . Now, pre-multiplying Z on both sides of (9), we have:

$$E_z \dot{\hat{x}}_i = A_z \hat{x}_i + [B_z \quad L_z] u_o, \quad (10)$$

where $E_z = ZE$, $A_z = ZA_l$, $B_z = ZB$ and $L_z = ZL$. Let the matrices A_z , B_z and L_z be written conformally with the matrix $E_z = [E_1^T \quad 0^T]^T$ as follows:

$$A_z = \begin{bmatrix} A_{z1} \\ A_{z2} \end{bmatrix}, \quad B_z = \begin{bmatrix} B_{z1} \\ B_{z2} \end{bmatrix}, \quad L_z = \begin{bmatrix} L_{z1} \\ L_{z2} \end{bmatrix}. \quad (11)$$

Let P and Q be two orthogonal matrices that satisfy the following SVD:

$$E_z = P \begin{bmatrix} \Sigma_q & 0 \\ 0 & 0 \end{bmatrix} Q^T, \quad E_z^+ = Q \begin{bmatrix} \Sigma_q^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^T, \quad (12)$$

where $\Sigma_q \in \mathbb{R}^{q \times q}$ is a diagonal matrix, with diagonal entries as the non-zero singular values of E_z . Let P and Q be partitioned as follows: $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$, $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$, where $P_{11} \in \mathbb{R}^{q \times q}$, $P_{12} \in \mathbb{R}^{q \times (n-q)}$, $P_{21} \in \mathbb{R}^{(n-q) \times q}$,

$P_{22} \in \mathbb{R}^{(n-q) \times (n-q)}$, $Q_1 \in \mathbb{R}^{n \times q}$ and $Q_2 \in \mathbb{R}^{n \times (n-q)}$. Using the partitioned matrices, define:

$$\begin{aligned} F_d &= \Sigma_q^{-1} P_{11}^T A_{z_1} (Q_1 - Q_2 (A_{z_2} Q_2)^{-1} A_{z_2} Q_1), \\ G_d &= \Sigma_q^{-1} P_{11}^T (B_{z_1} - A_{z_1} Q_2 (A_{z_2} Q_2)^{-1} B_{z_2}), \\ L_d &= \Sigma_q^{-1} P_{11}^T (L_{z_1} - A_{z_1} Q_2 (A_{z_2} Q_2)^{-1} L_{z_2}), \\ F_a &= -(A_{z_2} Q_2)^{-1} A_{z_2} Q_1, \quad G_a = -(A_{z_2} Q_2)^{-1} B_{z_2}, \\ L_a &= -(A_{z_2} Q_2)^{-1} L_{z_2}. \end{aligned} \quad (13)$$

Then, we have the following result for the reduced-order observer.

Theorem 1: A q^{th} (rank(E)) order observer for an impulse-free agent dynamics (6) is:

$$\begin{aligned} \dot{\hat{w}}_{d_i} &= F_d \hat{w}_{d_i} + G_d u_{s_i} + L_d y_i \\ \hat{w}_{a_i} &= F_a \hat{w}_{d_i} + G_a u_{s_i} + L_a y_i \end{aligned} \quad (14)$$

where $F_d \in \mathbb{R}^{q \times q}$, $G_d \in \mathbb{R}^{q \times m}$, $L_d \in \mathbb{R}^{q \times p}$, $F_a \in \mathbb{R}^{(n-q) \times q}$, $G_a \in \mathbb{R}^{(n-q) \times m}$ and $L_a \in \mathbb{R}^{(n-q) \times p}$ are as in (13). The estimated state of (6) is: $\hat{x}_i = Q \hat{w}_i$, where

$$\hat{w}_i = \begin{bmatrix} I \\ 0 \end{bmatrix} \hat{w}_{d_i} + \begin{bmatrix} 0 \\ I \end{bmatrix} \hat{w}_{a_i}. \quad (15)$$

Proof: The proof is given in the Appendix. ■

The estimated states $\hat{x}_i$ from q^{th} order observer are used to compute the control signal $u_{s_i}(t)$ according to (4) for achieving synchronization in the network. In the following section, we develop a procedure to design $u_{s_i}(t)$.

B. Distributed Controller Design Method

Define a matrix $\mathcal{L}_g$ as follows:

$$\mathcal{L}_g := \beta \mathcal{L} + \text{diag} \{ \gamma_1 h_1, \gamma_2 h_2, \dots, \gamma_r h_r \}, \quad (16)$$

where $\mathcal{L}$ is the Laplacian matrix of $G_s(\mathcal{V}', \mathcal{E}')$. Further, define the vectors: $\hat{\mathbf{x}} := [\hat{x}_1^T \hat{x}_2^T \dots \hat{x}_r^T]^T$, $\hat{\mathbf{x}}_0 := [\hat{x}_0^T \hat{x}_0^T \dots \hat{x}_0^T]^T$, $\hat{\zeta} := [\hat{\zeta}_1^T \hat{\zeta}_2^T \dots \hat{\zeta}_r^T]^T$. Then, using the fact that $(\mathcal{L} \otimes I_n) \hat{\mathbf{x}} = (\mathcal{L} \otimes I_n) (\hat{\mathbf{x}} - \hat{\mathbf{x}}_0)$, the relation in (5) can be represented as:

$$\hat{\zeta} := (\mathcal{L}_g \otimes I_n) (\hat{\mathbf{x}} - \hat{\mathbf{x}}_0). \quad (17)$$

Furthermore, by defining a vector $\mathbf{u}_s = [u_{s_1}^T u_{s_2}^T \dots u_{s_r}^T]^T$, the control law resulting from (4) is represented as follows:

$$\mathbf{u}_s = (I_r \otimes K) \hat{\zeta} = (\mathcal{L}_g \otimes K) (\hat{\mathbf{x}} - \hat{\mathbf{x}}_0). \quad (18)$$

Let $\mathbf{x} := [x_1^T x_2^T \dots x_r^T]^T$, $\mathbf{x}_0 := [x_0^T x_0^T \dots x_0^T]^T$. Then, following relations are obtained from (6):

$$(I_r \otimes E) \dot{\mathbf{x}}_0 = (I_r \otimes A_y) \mathbf{x}_0, \quad (19)$$

$$(I_r \otimes E) \dot{\mathbf{x}} = (I_r \otimes A_y) \mathbf{x} + (I_r \otimes B) \mathbf{u}_s. \quad (20)$$

Now, using (18), the follower dynamics (20) becomes:

$$(I_r \otimes E) \dot{\mathbf{x}} = (I_r \otimes A_y) \mathbf{x} + (\mathcal{L}_g \otimes BK) (\hat{\mathbf{x}} - \hat{\mathbf{x}}_0). \quad (21)$$

Let the synchronization error vectors be defined as follows: $\delta_i := x_i - x_0$, for all $i \in \mathbb{N}$. By defining a vector $\delta := [\delta_1^T \delta_2^T \dots \delta_r^T]^T$, we have: $\delta = \mathbf{x} - \mathbf{x}_0$. Then, with pre-multiplication of $(I_r \otimes E)$, and taking the derivatives on both sides of $\delta = \mathbf{x} - \mathbf{x}_0$, yield:

$$(I_r \otimes E) \dot{\delta} = (I_r \otimes E) \dot{\mathbf{x}} - (I_r \otimes E) \dot{\mathbf{x}}_0. \quad (22)$$

Define the following error vectors: $\tilde{x}_{d_i} = x_{d_i} - \hat{x}_{d_i}$ and $\tilde{x}_{a_i} = x_{a_i} - \hat{x}_{a_i}$. Then, by defining a vector $\tilde{\mathbf{x}}_i := \tilde{x}_{d_i} + \tilde{x}_{a_i}$, it can be observed that $\tilde{\mathbf{x}}_i = x_i - \hat{x}_i$, for all $i \in \mathbb{N}_0$, where $\hat{x}_i = \hat{x}_{d_i} + \hat{x}_{a_i}$ is the estimated state of i^{th} agent state x_i . Further, define $\tilde{\mathbf{x}}_0 = [\tilde{x}_0^T \tilde{x}_0^T \dots \tilde{x}_0^T]^T$ and $\tilde{\mathbf{x}} = [\tilde{x}_1^T \tilde{x}_2^T \dots \tilde{x}_r^T]^T$. Then, $\hat{\mathbf{x}} = \mathbf{x} - \tilde{\mathbf{x}}$ and $\hat{\mathbf{x}}_0 = \mathbf{x}_0 - \tilde{\mathbf{x}}_0$. Using (19) and (21), following synchronization error dynamics is obtained from (22):

$$\begin{aligned} (I_r \otimes E) \dot{\delta} &= [(I_r \otimes A_y) + (\mathcal{L}_g \otimes BK)] \delta - (\mathcal{L}_g \otimes BK) \tilde{\mathbf{x}} \\ &\quad + (\mathcal{L}_g \otimes BK) \tilde{\mathbf{x}}_0. \end{aligned} \quad (23)$$

Then, the following result holds.

Theorem 2: Consider the agent dynamics (6) with its reduced-order observer (14). Let for all $i \in \mathbb{N}$,

$$\text{eig}(F_d) \in \Omega_L, \quad \text{fnp}(E, A_y + \lambda_i(\mathcal{L}_g)BK) \in \Omega_K \quad (24)$$

where the regions Ω_L and Ω_K in the complex plane are as shown in Figure 1(c). Then, the following statements hold:

- $\lim_{t \rightarrow \infty} \|\hat{x}_i - x_i\|_2 = 0$, and the converges happens with a minimum decay rate α_2 ,
- $\lim_{t \rightarrow \infty} \|x_i - x_0\|_2 = 0$, and the converges happens with a minimum decay rate α_1 .

Proof: The proof is given in the Appendix. ■

According to Theorem 2, we can achieve state estimation and synchronization objectives by ensuring that $\text{eig}(F_d)$ and $\text{fnp}(E, A_y + \lambda_i(\mathcal{L}_g)BK)$, for all $i \in \mathbb{N}$, are in the open left half of the complex plane. Recall that the gain matrix L is designed such that the eigenvalues of F_d belong to Ω_L (refer to the last paragraph of the proof of Theorem 1). Hence, we only need to ensure that $\text{fnp}(E, A_y + \lambda_i(\mathcal{L}_g)BK)$, for all $i \in \mathbb{N}$, belong to the open left half of the complex plane. Note that the eigenvalues $\lambda_i(\mathcal{L}_g)$ are functions of β and γ_i , which are considered to be design parameters. Furthermore, the feedback gain matrix K is also a design parameter. We now use them to assign the finite poles of the pairs $(E, A_y + \lambda_i(\mathcal{L}_g)BK)$ in a stability region Ω_K , as shown in Figure 1(c).

Consider the orthogonal matrix Z , used in (10), and obtain: $ZE = [E_1^T \ 0^T]^T$, $ZA_y = [A_{zy_1}^T \ A_{zy_2}^T]^T$ and $ZB = [B_{z_1}^T \ B_{z_2}^T]^T$. Consider the orthogonal matrices P and Q , used in (12) with appropriate partitions (P_{ij} and Q_i). Using these matrices, define the following matrices:

$$\begin{aligned} A_{y_{11}} &:= \Sigma_q^{-1} P_{11}^T (A_{zy_1} Q_1 - A_{zy_1} Q_2 (A_{zy_2} Q_2)^{-1} A_{zy_2} Q_1), \\ B_{y_1} &:= \Sigma_q^{-1} P_{11}^T (B_{z_1} - A_{zy_1} Q_2 (A_{zy_2} Q_2)^{-1} B_{z_2}), \\ K &:= [K_1 \ 0] Q^T, \end{aligned} \quad (25)$$

where $A_{y_{11}} \in \mathbb{R}^{q \times q}$, $B_{y_1} \in \mathbb{R}^{q \times p}$ and $K_1 \in \mathbb{R}^{p \times q}$. Then, the following relation holds, which follows from [35, Theorem 3]: $\bigcup_{i=1}^r \text{fnp}(E, A_y + \lambda_i(\mathcal{L}_g)BK) = \bigcup_{i=1}^r \text{eig}(A_{y_{11}} + \lambda_i(\mathcal{L}_g)B_{y_1}K_1)$. Hence, we use the design parameters β , γ_i and K_1 to place the eigenvalues of $A_{y_{11}} + \lambda_i(\mathcal{L}_g)B_{y_1}K_1$ in Ω_K through the following result.

Theorem 3: Assume that the matrix K_1 and symmetric positive definite (SPD) matrices: X and S satisfy the following matrix equation:

$$A_{y_{11}} X + X A_{y_{11}}^T + M + 2\alpha_1 X = -S. \quad (26)$$

where $M = B_{y_1}K_1X + XK_1^TB_{y_1}^T$, and α_1 is a positive scalar, which is used to define Ω_K , as in Figure 1(a). Using the solution X , S and K_1 of (26), select the scalars:

$$\gamma_i = \frac{1}{h_i}, \quad \forall i \in \mathbb{N}; \quad 0 < \beta < \frac{\lambda_{\min}(S)}{\rho(\mathcal{L})\|M\|_2}. \quad (27)$$

Then, $\bigcup_{i=1}^r \text{eig}(A_{y_{11}} + \lambda_i(\mathcal{L}_g)B_{y_1}K_1) \subset \Omega_K$, for all $i \in \mathbb{N}$.

Proof: To prove this result, we first consider a feedback gain matrix as follows: $K_1(\psi) := K_1 + \psi K_1$, where ψ is considered to be an uncertain real parameter. Since K_1 satisfies the matrix equation (26), it follows from [36] that the eigenvalues of $(A_{y_{11}} + B_{y_1}K_1)$ are contained within the stability region Ω_K . Then, our aim is to find a specific range $\mathcal{R}$ for the uncertain parameter ψ such that the following set relation is satisfied:

$$\text{eig}(A_{y_{11}} + B_{y_1}K_1 + \psi B_{y_1}K_1) \subset \Omega_K, \quad \forall \psi \in \mathcal{R}. \quad (28)$$

It is shown in the proof of [7, Theorem 2] that for every ψ satisfying: $|\psi| < \frac{\lambda_{\min}(S)}{\|M\|_2}$, following matrix inequality holds:

$$A_{y_{11}}X + XA_{y_{11}}^T + (1 + \psi)M + 2\alpha_1X \prec 0. \quad (29)$$

Hence, by setting the range $\mathcal{R} = \left(-\frac{\lambda_{\min}(S)}{\|M\|_2}, \frac{\lambda_{\min}(S)}{\|M\|_2}\right)$, it follows from [36] that the relation (28) holds for all $\psi \in \mathcal{R}$.

Recall that the eigenvalues of the Laplacian matrix $\lambda_i(\mathcal{L})$, for $i \in \mathbb{N}$, could be any real number, since the edge weights w_{ij} of the network graph are both positive or negative. Then, according to the definition of $\mathcal{L}_g$, as in (16), and for the choice of γ_i , as in (27), the eigenvalues of $\mathcal{L}_g$ are: $\lambda_i(\mathcal{L}_g) = 1 + \beta\lambda_i(\mathcal{L})$. Since the gain parameter β satisfies (27), we have: $\beta\rho(\mathcal{L}) < \frac{\lambda_{\min}(S)}{\|M\|_2}$. Then, for all $i \in \mathbb{N}$, since β is a positive real number, and $\lambda_i(\mathcal{L})$ is any real number, it follows that: $|\beta\lambda_i(\mathcal{L})| \leq \beta\rho(\mathcal{L}) < \frac{\lambda_{\min}(S)}{\|M\|_2}$, which implies: $\beta\lambda_i(\mathcal{L}) \in \mathcal{R} = \left(-\frac{\lambda_{\min}(S)}{\|M\|_2}, \frac{\lambda_{\min}(S)}{\|M\|_2}\right)$, $\forall i \in \mathbb{N}$.

It is already shown that for all $\psi \in \mathcal{R}$, the matrix inequality (29) holds. Hence, by setting the uncertain parameter $\psi = \beta\lambda_i(\mathcal{L})$, for all $i \in \mathbb{N}$, the following set of matrix inequalities hold, since (29) holds:

$$A_{y_{11}}X + XA_{y_{11}}^T + (1 + \beta\lambda_i(\mathcal{L}))M + 2\alpha_1X \prec 0. \quad (30)$$

This implies $\bigcup_{i=1}^r \text{eig}(A_{y_{11}} + \lambda_i(\mathcal{L}_g)B_{y_1}K_1) \subset \Omega_K$, which follows from [36]. The proof is completed. ■

Note that Theorem 3 provides a procedure to compute the gain matrix K_1 , scalar gains β and γ_i . To obtain K_1 , one needs to solve the matrix equation (26). However, (26) is not linear in variables K_1 and X . Hence, a new matrix variable $Y := K_1X$ is defined, and the following LMI feasibility problem is formulated.

Problem 2: For a given α_1 , find X , Y and S such that:

- i) $X \succ 0$, ii) $S \succ 0$,
- iii) $A_{y_{11}}X + B_{y_1}Y + XA_{y_{11}}^T + Y^TB_{y_1}^T + 2\alpha_1X = -S$.

We have assumed that the agent dynamics are C -controllable. Hence, it follows from [35, Theorem 4] that the pair $(A_{y_{11}}, B_{y_1})$ is controllable, and hence, Problem 2

has a feasible solution. After computing a feasible solution of Problem 2, the matrix K_1 can be obtained as follows: $K_1 = YX^{-1}$. Then, according to the definition as in (25): $K = [K_1 \ 0]Q^T$. Further, the other gain parameters β and γ_i are then chosen according to (27) using the solution of Problem 2. The resulting K , β and γ_i , will ensure that $\text{fnp}(E, A_y + \lambda_i(\mathcal{L}_g)BK) \subset \Omega_K$, for all $i \in \mathbb{N}$. Hence, follower states $x_i(t)$ synchronize with the leader state $x_0(t)$ with a minimum decay rate α_1 . In the forthcoming result, we show that the implementation of the above-discussed reduced-order observer with a distributed controller will ensure that the closed-loop system is impulse-free. For this, we define the following vectors:

$$\begin{aligned} \hat{\mathbf{w}}_{d_0} &= [\hat{w}_{d_0}^T \ \hat{w}_{d_0}^T \ \cdots \ \hat{w}_{d_0}^T]^T, \quad \hat{\mathbf{w}}_d = [\hat{w}_{d_1}^T \ \hat{w}_{d_2}^T \ \cdots \ \hat{w}_{d_r}^T]^T, \\ \hat{\mathbf{w}}_{a_0} &= [\hat{w}_{a_0}^T \ \hat{w}_{a_0}^T \ \cdots \ \hat{w}_{a_0}^T]^T, \quad \hat{\mathbf{w}}_a := [\hat{w}_{a_1}^T \ \hat{w}_{a_2}^T \ \cdots \ \hat{w}_{a_r}^T]^T, \\ \mathbf{y}_0 &= [y_0^T \ y_0^T \ \cdots \ y_0^T]^T, \quad \mathbf{y} = [y_1^T \ y_2^T \ \cdots \ y_r^T]^T. \end{aligned}$$

Using these vectors, we have the following relations, which are obtained from (14):

$$\begin{aligned} \dot{\hat{\mathbf{w}}}_{d_0} &= (I_r \otimes F_d)\hat{\mathbf{w}}_{d_0} + (I_r \otimes G_d)\mathbf{u}_{s_0} + (I_r \otimes L_d)\mathbf{y}_0, \\ \dot{\hat{\mathbf{w}}}_{a_0} &= (I_r \otimes F_a)\hat{\mathbf{w}}_{a_0} + (I_r \otimes G_a)\mathbf{u}_{s_0} + (I_r \otimes L_a)\mathbf{y}_0, \\ \dot{\hat{\mathbf{w}}}_d &= (I_r \otimes F_d)\hat{\mathbf{w}}_d + (I_r \otimes G_d)\mathbf{u}_s + (I_r \otimes L_d)\mathbf{y}, \\ \dot{\hat{\mathbf{w}}}_a &= (I_r \otimes F_a)\hat{\mathbf{w}}_a + (I_r \otimes G_a)\mathbf{u}_s + (I_r \otimes L_a)\mathbf{y}. \end{aligned} \quad (31)$$

Further, it follows from (4) and (18) that: $\mathbf{u}_{s_0} = 0$, and $\mathbf{u}_s = (\mathcal{L}_g \otimes K)(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0)$. Then, the following relations are obtained from (31) using: $\mathbf{y}_0 = (I_r \otimes C)\mathbf{x}_0$, $\mathbf{y} = (I_r \otimes C)\mathbf{x}$:

$$\begin{aligned} \dot{\hat{\mathbf{w}}}_{d_0} &= (I_r \otimes F_d)\hat{\mathbf{w}}_{d_0} + (I_r \otimes L_dC)\mathbf{x}_0, \\ \dot{\hat{\mathbf{w}}}_{a_0} &= (I_r \otimes F_a)\hat{\mathbf{w}}_{a_0} + (I_r \otimes L_aC)\mathbf{x}_0, \\ \dot{\hat{\mathbf{w}}}_d &= (I_r \otimes F_d)\hat{\mathbf{w}}_d + (\mathcal{L}_g \otimes G_dK)(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0) + (I_r \otimes L_dC)\mathbf{x}, \\ \dot{\hat{\mathbf{w}}}_a &= (I_r \otimes F_a)\hat{\mathbf{w}}_a + (\mathcal{L}_g \otimes G_aK)(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0) + (I_r \otimes L_aC)\mathbf{x}. \end{aligned} \quad (32)$$

In addition, the relations (15) and the fact that $\hat{x}_i = Q\hat{w}_i$, for $i = 0, 1, \dots, r$, yield the following two algebraic equations:

$$\begin{aligned} \hat{\mathbf{x}}_0 &= (I_r \otimes Q_1)\hat{\mathbf{w}}_{d_0} + (I_r \otimes Q_2)\hat{\mathbf{w}}_{a_0}, \\ \hat{\mathbf{x}} &= (I_r \otimes Q_1)\hat{\mathbf{w}}_d + (I_r \otimes Q_2)\hat{\mathbf{w}}_a. \end{aligned} \quad (33)$$

where $Q = [Q_1 \ Q_2]$. The leader dynamics and follower dynamics are as in (19) and (21), respectively. Hence, the following set of differential and algebraic equations: (19), (21), (32), and (33) constitute the closed-loop system. Then, by defining the closed-loop state vector as: $\mathbf{x}_{cl} := [\mathbf{x}_0^T \ \hat{\mathbf{w}}_{d_0}^T \ \hat{\mathbf{w}}_{a_0}^T \ \hat{\mathbf{x}}_0^T \ \hat{\mathbf{x}}^T \ \mathbf{x}^T \ \hat{\mathbf{w}}_d^T \ \hat{\mathbf{w}}_a^T]^T$, the closed-loop dynamics can be written as:

$$E_{cl}\dot{\mathbf{x}}_{cl} = A_{cl}\mathbf{x}_{cl}, \quad (34)$$

where, E_{cl} and A_{cl} are given as in (†) (refer to Appendix C). Now, we have the following result.

Theorem 4: The state response of the closed-loop system (34) is impulse-free.

Proof: The proof is given in Appendix C. ■

Remark 1: It is worth noting that despite the presence of negative edge weights in the network graph, we could achieve leader-following synchronization in the closed loop. This is primarily due to the systematic design of the gain

parameters: K_1 , β , and γ_i through *parametric robust control* framework. A major advantage of this framework is its unified capability to accommodate both positive and negative edge weights of the Laplacian matrix while ensuring the closed-loop synchronization. This robust design method also guarantees leader-following synchronization even in the extreme case where all the edge weights of the network graph are negative. Essentially, the designed network gains modify the underlying network weights (without changing the network topology) in such a way that the synchronization is enforced in the closed loop despite antagonistic interactions.

C. Leader-follower synchronization under uncertain communication network

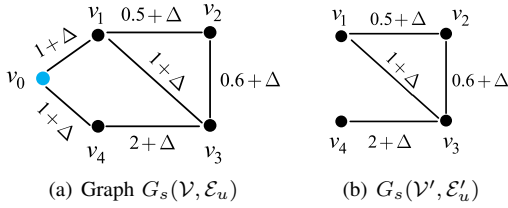


Fig. 2. Uncertain network graph in (a) with its subgraph in (b).

The control algorithm developed in the previous section is over a fixed and fully known communication network, that is, the exact edge weights of the graph are known. We now extend the developed results to address the leader-following synchronization problem in an uncertain communication network. Denote the uncertain network graph as $G_s(\mathcal{V}, \mathcal{E}_u)$, where $\mathcal{E}_u$ is the set of edges representing the uncertain communication channel between the agents. Similar to [17], each uncertain communication channel is represented by an additive uncertain edge weight: $\tilde{w}_{ij} := w_{ij} + \Delta$, where w_{ij} is the nominal weight, and Δ is the uncertain parameter, which can take any real number in the interval: $[\Delta^-, \Delta^+]$. Such an uncertain graph is depicted in Figure 2 (a). We compute a range $[\Delta^-, \Delta^+]$ for Δ such that for all $\Delta \in [\Delta^-, \Delta^+]$, the impulse elimination and prescribed decay rate leader-follower synchronization objectives are achieved. For this, we consider the feedback control law $u_i(t)$ of the form (2) with $u_{m_i}(t)$ as in (3) and $u_{s_i}(t)$ as in (4). The signal $\hat{\zeta}_i(t)$ in (4), which is based on neighborhood estimated relative states of leader and follower agents, however, is now defined as follows:

$$\hat{\zeta}_i = \sum_{j \in \mathcal{N}_i} \beta (\tilde{w}_{ij}(\hat{x}_i - \hat{x}_j)) + \gamma_i (\tilde{h}_i(\hat{x}_i - \hat{x}_0)), \quad (35)$$

where β and γ_i are the *network gain parameters*, which need to be designed, and $\tilde{w}_{ij} = w_{ij} + \Delta$ is the uncertain edge weight in the network subgraph $G_s(\mathcal{V}', \mathcal{E}'_u)$ as in Figure 2 (b). The quantity $\tilde{h}_i$ in (35) is defined as follows, similar to $h_i = w_{ij}$ as defined earlier:

$$\tilde{h}_i = w_{ij} + \Delta, \text{ for } j \in \mathcal{N}_i. \quad (36)$$

Using the definitions of $\hat{\mathbf{x}}$, $\hat{\mathbf{x}}_0$ and $\hat{\zeta}$, introduced in Subsection III-B, (35) can compactly be represented as:

$$\hat{\zeta} = (\mathcal{L}_h \otimes I_n) (\hat{\mathbf{x}} - \hat{\mathbf{x}}_0). \quad (37)$$

The matrix $\mathcal{L}_h$ in (37) is of the following form, which is obtained by using the definitions of uncertain parameters $\tilde{w}_{ij} = w_{ij} + \Delta$ and $\tilde{h}_i$ as in (36):

$$\mathcal{L}_h := \beta \mathcal{L} + \beta \Delta \mathcal{L}_\delta + H_\gamma + \Delta H_\delta, \quad (38)$$

where $\mathcal{L}$ and $\mathcal{L}_\delta$ are the Laplacian matrices of the subgraph $G_s(\mathcal{V}', \mathcal{E}'_u)$ with respect to the nominal edge weights w_{ij} and edge weights equal to 1, respectively, the matrices: $H_\gamma = \text{diag}\{\gamma_1 h_1, \gamma_2 h_2, \dots, \gamma_r h_r\}$ and $H_\delta = \text{diag}\{\gamma_1, \gamma_2, \dots, \gamma_r\}$. For instance, corresponding to the network graph $G_s(\mathcal{V}', \mathcal{E}'_u)$ as shown in Figure 2, the matrices in (38) are:

$$\mathcal{L} = \begin{bmatrix} 1.5 & -0.5 & -1 & 0 \\ -0.5 & 1.1 & -0.6 & 0 \\ -1 & -0.6 & 3.6 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}, \quad \mathcal{L}_\delta = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

$$H_\gamma = \text{diag}\{\gamma_1; 0.6\gamma_2; 2\gamma_3; \gamma_4\}, \quad H_\delta = \text{diag}\{\gamma_1; \gamma_2; \gamma_3; \gamma_4\}.$$

To address the considered problem, the setup developed in the previous section will remain the same, except for the network connectivity information. Based on the definition of neighborhood estimated relative signal $\hat{\zeta}_i$ in (35), the network associated matrix is now $\mathcal{L}_h$, as defined in (38), instead of $\mathcal{L}_g$. Hence, the changes required to achieve the control objectives are in the design of the feedback gain matrix K and the network gain parameters β and γ_i . For this, we use Theorem 3, and through the following result, we first design the gain matrix K , network gains β and γ_i , and then, compute an allowable bound for the uncertain parameter Δ .

Theorem 5: Let the matrix K_1 , and SPD matrices: X and S satisfy the matrix equation (26) as in Theorem 3. Let $\psi = \frac{\lambda_{\min}(S)}{\|M\|_2}$, $\gamma_i = \frac{1}{h_i}$ and $0 < \beta < \frac{\psi}{\rho(\mathcal{L})}$. Further, assume that $\mathcal{L}_\delta$ and H_δ be as in (38). Then, for any values of Δ satisfying:

$$|\Delta| < \frac{\psi - \beta \rho(\mathcal{L})}{\beta \rho(\mathcal{L}_\delta) + \rho(H_\delta)}, \quad (39)$$

following holds: $\bigcup_{i=1}^r \text{eig}(A_{y_{11}} + \lambda_i(\mathcal{L}_h)B_{y_1}K_1) \subset \Omega_K$.

Proof: It can be observed from the definition of $\mathcal{L}_h$ in (38) that for the choice of scalar gain parameter $\gamma_i = \frac{1}{h_i}$, the eigenvalues of $\mathcal{L}_h$ are:

$$\lambda_i(\mathcal{L}_h) = 1 + \lambda_i(\beta \mathcal{L} + \beta \Delta \mathcal{L}_\delta + \Delta H_\delta). \quad (40)$$

We now compute an upper bound for the largest singular value of the matrix: $\mathcal{L}_{h\delta} := \beta \mathcal{L} + \beta \Delta \mathcal{L}_\delta + \Delta H_\delta$. Using [37, Theorem 3.3.16] on the singular values of the addition of symmetric matrices, we have the following relation:

$$\begin{aligned} \sigma_{\max}(\mathcal{L}_{h\delta}) &\leq \sigma_{\max}(\beta \mathcal{L}) + \sigma_{\max}(\beta \Delta \mathcal{L}_\delta) + \sigma_{\max}(\Delta H_\delta) \\ &\leq |\sigma_{\max}(\beta \mathcal{L})| + |\sigma_{\max}(\beta \Delta \mathcal{L}_\delta)| + |\sigma_{\max}(\Delta H_\delta)| \\ &\leq |\sigma_{\max}(\beta \mathcal{L})| + |\beta| |\Delta| |\sigma_{\max}(\mathcal{L}_\delta)| \\ &\quad + |\Delta| |\sigma_{\max}(H_\delta)|. \end{aligned} \quad (41)$$

Note that according to the choice the scalar gain β is positive, and $\sigma_{\max}(\bullet) > 0$. Hence, the relation in (41) becomes:

$$\sigma_{\max}(\mathcal{L}_{h\delta}) \leq \beta \sigma_{\max}(\mathcal{L}) + \beta |\Delta| \sigma_{\max}(\mathcal{L}_\delta) + |\Delta| \sigma_{\max}(H_\delta). \quad (42)$$

Since $\mathcal{L}$ is a symmetric matrix, we have: $\sigma_i(\mathcal{L}) = |\lambda_i(\mathcal{L})|$ and $\sigma_{\max}(\mathcal{L}) = \max_i |\lambda_i(\mathcal{L})| = \rho(\mathcal{L})$. Similar relations also hold

for matrices $\mathcal{L}_\delta$ and H_δ . Then, the relation in (42) can be represented as:

$$\max_i |\lambda_i(\mathcal{L}_{h\delta})| \leq \beta \rho(\mathcal{L}) + \beta |\Delta| \rho(\mathcal{L}_\delta) + |\Delta| \rho(H_\delta). \quad (43)$$

Further, we have: $\lambda_i(\mathcal{L}_{h\delta}) \leq \max_i |\lambda_i(\mathcal{L}_{h\delta})|$. Hence, it follows from (43), and the relation for $|\Delta|$ in (39), that for $i \in \mathbb{N}$:

$$\lambda_i(\mathcal{L}_{h\delta}) \leq \beta \rho(\mathcal{L}) + \beta |\Delta| \rho(\mathcal{L}_\delta) + |\Delta| \rho(H_\delta) < \psi. \quad (44)$$

For every $i \in \mathbb{N}$, following relation holds, which follows from (40): $A_{y11} + \lambda_i(\mathcal{L}_h)B_{y1}K_1 = A_{y11} + (1 + \lambda_i(\mathcal{L}_{h\delta}))B_{y1}K_1$. Since (26) holds, it follows from the proof of Theorem 3 that for every ψ satisfying: $|\psi| < \frac{\lambda_{\min}(S)}{\|M\|_2}$, the eigenvalues of: $A_{y11} + (1 + \psi)B_{y1}K_1$ belong to the stability region Ω_K . It then follows from (44) that $\bigcup_{i=1}^r \text{eig}(A_{y11} + \lambda_i(\mathcal{L}_h)B_{y1}K_1) \subset \Omega_K$. Hence, the result holds. ■

Remark 2: Theorem 5 provides an approach to select the feedback gain matrix K_1 and network gain parameters β and γ_i to achieve leader-following synchronization in an SMAS under an uncertain communication network. In addition, it gives an upper bound for the uncertain parameter $|\Delta|$ such that for $\Delta \in [\Delta^-, \Delta^+]$ the control objectives are achieved.

To compute the feedback gain matrix K and parameter ψ , we formulate an LMI optimization to enlarge $|\Delta|$. Observe that Δ depends on ψ , and ψ depends on $\lambda_{\min}(S)$ and $\|M\|_2$. Hence, to enlarge $|\Delta|$, we maximize the smallest eigenvalue of S and minimize $\|M\|_2$ through the following optimization.

Problem 3: For a given α_1 , corresponding to the desired synchronization speed, solve:

$$\begin{aligned} & \max (\eta - \varrho) \quad \text{subject to} \\ & i) X \succ 0, \quad ii) S - \eta I \succ 0, \quad iii) \begin{bmatrix} \varrho I & M \\ M^T & \varrho I \end{bmatrix} \succ 0, \\ & iv) A_{y11}X + XA_{y11}^T + M + 2\alpha_1 X = -S, \end{aligned}$$

for $\eta > 0$ and $M = B_{y1}Y + Y^TB_{y1}^T$.

Once ψ is computed by solving Problem 3, one needs to select β and γ_i according to Theorem 5.

IV. DEMONSTRATIVE EXAMPLES

Example 1: Consider an SMAS of the form (1), consisting of seven agents with system matrices as follows:

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & 5 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}^T.$$

The communication network among the agents is as per the graph in Figure 3. The edge weights are: $w_{13} = w_{31} = -0.3$, $w_{14} = w_{41} = 0.9$, $w_{24} = w_{42} = -0.2$, $w_{34} = w_{43} = 0.8$, $w_{35} = w_{53} = -1$, $w_{45} = w_{54} = 0.3$, $w_{56} = w_{65} = -0.8$, and edge weight from leader to neighborhood followers are: $w_{01} = 1$, $w_{02} = 1$. To compute the associated gain parameters, we follow the procedure discussed in Section III.

For the given system matrices, we have $q = \text{rank}(E) = 2$, and it (pair (E, A)) has only one finite pole. Hence, the state response of the considered system is impulsive [11, Theorem 7.1]. To eliminate impulse, we use the output feedback control as in (3), for which the associated feedback gain matrix is chosen as follows: $K_m = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, according to the procedure

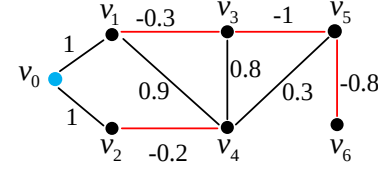


Fig. 3. A networked graph with cooperative and antagonistic interaction (v_0 : leader agent, v_i : for $i = \{1, 2, \dots, 6\}$, follower agents).

given in [11, Chapter 11]. With this K_m , it is verified that the finite poles of the pair $(E, A + BK_mC)$ are located at $-1.2 \pm 2.9933i$. For the neighborhood estimation signal $\hat{z}$ as in (5), we choose the weights h_i as follows: $h_1 = w_{01} = 1$, $h_2 = w_{02} = 1$, $h_3 = w_{34} = 0.8$, $h_4 = w_{14} = 0.9$, $h_5 = w_{35} = -1$, $h_6 = w_{56} = -0.8$.

To compute the gain matrix K , we follow the procedure given in Section III-B. Moreover, since $\text{fnp}(E, A_y - LC) = \text{fnp}(E^T, A_y^T - C^T L^T)$, using the concept of duality, we design observer gain matrix L similar to the design of K . For this, the system matrices are set as: $E = E^T$, $A_y = A_y^T$, $B = C^T$, and $K = L^T$. Then, the associated feasibility problems are solved with an LMI solver *SeDuMi* [38], by choosing $\alpha_1 = 2$ (for Ω_K) and $\alpha_2 = 8$ (for Ω_L). The computed gain matrices are: $K = \begin{bmatrix} 0 & 5.5553 & 0 & -3.5784 \\ 0 & 3.4947 & 0 & -1.8684 \end{bmatrix}$, $L = \begin{bmatrix} 0.0256 & -1.1276 & 0 & 0 \\ 9.4372 & 2.2520 & 0 & 0 \end{bmatrix}^T$.

The following parameters are computed from the solution of Problem 2: $\lambda_{\min}(S) = 0.3296$ and $\|M\|_2 = 1.2050$. The spectral radius of matrix $\mathcal{L}$ is: $\rho(\mathcal{L}) = 2.6056$. Then, we choose $\gamma_i = \frac{1}{h_i}$ and $\beta = 0.1049$ according to (27). Further, considering the procedure given in Subsection III-A, we compute the following matrices for the reduced-order observer according to (13):

$$F_d = \begin{bmatrix} -7.2423 & -13.1401 \\ 3.3736 & -11.1370 \end{bmatrix}, G_d = \begin{bmatrix} 0 & -0.9744 \\ 1 & -3.1276 \end{bmatrix}, G_a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ L_d = \begin{bmatrix} -0.0256 & -9.4372 \\ 1.1276 & -2.2520 \end{bmatrix}, F_a = \begin{bmatrix} -0.2 & 3.8 \\ -0.2 & -0.2 \end{bmatrix}, L_a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

With the above computed gain parameters and reduced-order observer, the closed-loop SMAS is simulated in MATLAB Simulink (Version R2022a) by obtaining a set of consistent initial conditions. The error signals between the estimated states and actual states are shown in Figure 4, which asymptotically goes to zero, satisfying the decay rate $\alpha_2 = 8$. Further, the states of leader and follower agents are depicted in Figure 5. It can be observed that the follower states converge to the leader states, satisfying a decay rate $\alpha_1 = 2$ (within $t_s = \frac{4}{2} = 2$ seconds). To demonstrate the effect of decay rate on synchronization speed, we next choose $\alpha_1 = 8$. Then we compute the following control gain parameters: $K = \begin{bmatrix} 0 & 22.9195 & 0 & -11.6379 \\ 0 & 10.1480 & 0 & -2.2069 \end{bmatrix}$, $\beta = 0.0114$. The corresponding simulation results are presented in Figure 6, where it can be noticed that the follower states converge to the leader within 0.5 ($t_s = \frac{4}{8}$) seconds. This demonstrates how the prescribed decay rate regulates the speed of synchronization.

Example 2: In this example, we consider a networked microgrid system, consisting of five identical microgrids (Fig. 7-(a)), with one leader and four followers. Each microgrid, represented in Fig. 7-(b), consists of two small-scale hydropower plants (G_1, G_2) and a local network with two

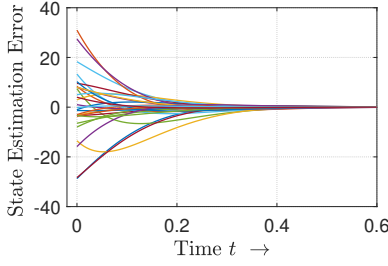


Fig. 4. Estimation error of all agents in the network: $(x_{ij} - \hat{x}_{ij}) \forall i = \{0, 1, \dots, 6\}, j = \{1, 2, 3, 4\}$.

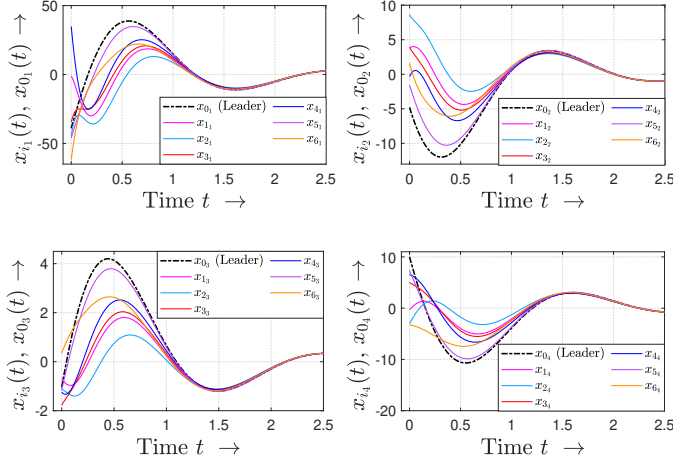


Fig. 5. Synchronization of follower states x_{ij} (solid lines) with leader state x_{0j} (dotted lines) for each $i = 1, \dots, 6$ and $j = 1, \dots, 4$ for $\alpha_1 = 2$ in Example 1, where x_{ij} and x_{0j} are the j^{th} state component of i^{th} follower and leader agent, respectively.

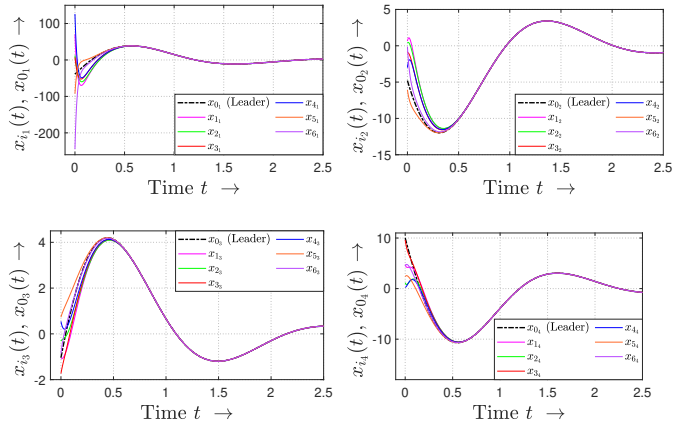


Fig. 6. Synchronization of follower states x_{ij} (solid lines) to leader state x_{0j} (dotted lines) $\forall i = 1, \dots, 6$ and $j = 1, \dots, 4$ for $\alpha_1 = 8$ in Example 1.

generator buses (1, 2), one load bus (3), and transmission lines (TL-1, TL-2, TL-3) connecting all buses. The linearized model of each microgrid is adopted from [39], assuming the following parameters: $M_1 = 0.0127$ seconds²/radian, $M_2 = 0.0190$ seconds²/radian, $D_1 = 0.0145$ pu (per unit), $D_2 = 0.0121$ pu, $X_{d1} = 0.3920$ pu, $X_{d2} = 0.2859$ pu, $B_{12} = 9.7840$ pu, $B_{13} = 5.9760$ pu and $B_{23} = 5.5880$ pu.

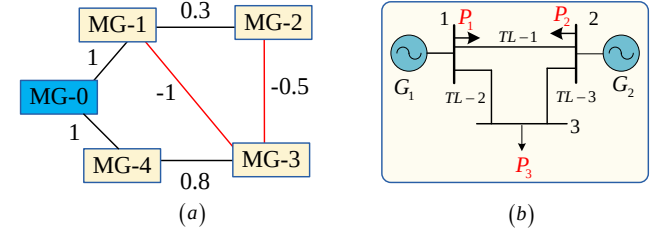


Fig. 7. (a) represents a network topology of 5 microgrids with MG-0 as the leader and MG- i for $i = 1, \dots, 4$ as followers. Figure (b) is a representation of each microgrid (MG) with two generators: G_1 and G_2 , three buses: 1, 2, and 3, and three transmission lines: TL-1, TL-2, and TL-3.

Then, the linearized model of each microgrid is represented by an SS of the form (1), where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0127 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0190 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -3.4977 & 0 & -0.0145 & 0 & 2.551 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -3.4977 & 0 & 0 & 0 & 18.311 & -9.784 \\ 0 & 0 & 0 & -9.784 & 18.8697 & -5.588 \\ 0 & 0 & 0 & -5.976 & -5.588 & 11.564 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since the individual microgrid is represented by an SS, the considered network of microgrids is an SMAS, whose underlying network topology is depicted in Figure 7-(a). In the considered network, the interaction between microgrid 1 with 3, and microgrid 2 with 3 are antagonistic in nature, and hence, the associated edges have negative edge weight. This antagonistic behavior may be due to transmission lines with negative reactances, or it could be related to cyber-attacks or unusual power-sharing trends between microgrid 3 and microgrids 1 and 2.

Considering the system matrices E and A , it is verified that the state response of individual microgrids in the open-loop condition is impulse-free. Hence, we take the gain matrix K_m as zero. Further, we obtain that the finite poles of individual microgrids are at $-0.4696 \pm j13.1645$, -0.8394 . To achieve leader-follower synchronization in the network, we construct the control signal $u_{s_i}(t)$, as defined in (4), for the follower microgrids. By choosing the stability regions Ω_K , with $\alpha_1 = 4$, and Ω_L with $\alpha_2 = 8$, we solve LMI feasibility problems to obtain the gain matrix K and L , which are as follows:

$$K = \begin{bmatrix} 0.937 & -0.277 & 0.326 & 0 & 0 & 0 \\ -0.471 & 0.164 & -0.259 & 0 & 0 & 0 \\ 0.208 & -0.048 & 0.023 & 0 & 0 & 0 \\ 0.061 & -0.002 & -0.038 & 0 & 0 & 0 \\ 0.137 & -0.026 & -0.006 & 0 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 2.8710 & -2.7784 \\ 0.3875 & -0.2438 \\ -0.3446 & 0.4701 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We follow the procedure discussed in Subsection III-A, and obtain the following matrices associated with the reduced-order observer as in (14):

$$F_d = \begin{bmatrix} 0 & -3.778 & 3.871 \\ 69.543 & -25.378 & 18.139 \\ -104.041 & 19.199 & -31.657 \end{bmatrix}, \quad L_d = \begin{bmatrix} -2.8710 & 2.7784 \\ -18.1392 & 24.742 \\ 30.5156 & -19.199 \end{bmatrix},$$

$$G_d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -52.63 & -27.26 & -32.75 & -29.91 \\ -78.74 & 0 & -37.95 & -29.74 & -33.98 \end{bmatrix}, \quad L_a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$F_a = \begin{bmatrix} -0.5683 & 0 & 0 \\ -0.6222 & 0 & 0 \\ -0.518 & 0 & 0 \end{bmatrix}, \quad G_a = \begin{bmatrix} 0 & -0.17 & -0.169 & -0.25 \\ 0 & -0.15 & -0.18 & -0.16 \\ 0 & -0.19 & -0.15 & -0.17 \end{bmatrix}.$$

Further, we compute the spectral radius of the Laplacian matrix as $\rho(\mathcal{L}) = 2.0032$, the scalar gain β is chosen as $\beta = 0.0023$, according to (27). In addition, we choose $\gamma_i = 1/h_i$, $\forall i \in \mathbb{N}$. With these parameters, the closed-loop SMAS is simulated in MATLAB, and the states of the leader and follower microgrids are depicted in Figure 8. It can be observed from the simulated results that the states of all the follower microgrids are synchronized with the state of the leader microgrid within $t_s = \frac{4}{4} = 1$ seconds (corresponds to the minimum decay rate of $\alpha_1 = 4$), without showing any impulsive behavior.

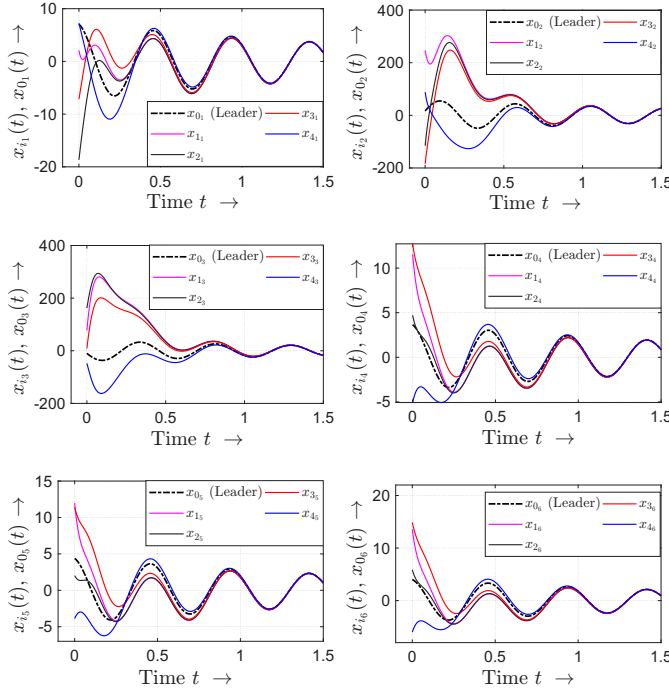


Fig. 8. Synchronization of follower states x_{ij} (solid lines) with leader state x_{0j} (dotted lines) for each $i = 1, \dots, 4$ and $j = 1, \dots, 6$ with $\alpha_1 = 4$ in Example 2, where x_{ij} and x_{0j} are the j^{th} state component of i^{th} follower and leader agent, respectively.

Example 3: In this example, we consider the same SMAS as in Example 1. However, we now consider the communication network graph in Figure 3 to be uncertain, that is, the edge weights are of the form $\tilde{w}_{ij} = w_{ij} + \Delta$. The nominal values of the edge weights w_{ij} and parameter h_i , used in (36), are considered the same as in Example 1. To achieve the desired control objectives, we use the same reduced-order observer. We solve the optimization Problem 3 and obtain the feedback gain matrix: $K = \begin{bmatrix} 0 & 9.0738 & 0 & -4.9364 \\ 0 & 3.2395 & 0 & -1.6041 \end{bmatrix}$.

With respect to the numerical data, we obtain $\lambda_{\min}(S) = 1.5953$ and $\|M\|_2 = 2.7340$. Then, we have computed $|\psi| = 0.5835$ (that is, $\mathcal{R} = (-0.5835, 0.5835)$). We choose the gain parameter: $\beta = 0.002$, and then, according to the relation (39), the uncertainty bound on network weights is: $|\Delta| < 0.46$, that is, $\Delta^- = -0.45$ and $\Delta^+ = 0.45$. For different values of $\Delta \in [\Delta^-, \Delta^+]$, the eigenvalues of the matrix $\mathcal{L}_{h\delta}$, as defined in Subsection III-C, are plotted, and is shown in Figure 9. It is clear that all the eigenvalues of $\mathcal{L}_{h\delta}$ lie within the range $\mathcal{R}$. Further, the finite poles of the pair

$((I_r \otimes E), (I_r \otimes A_y) + \mathcal{L}_h \otimes BK)$ are plotted in Figure 10-(a) for different values of $\Delta \in [\Delta^-, \Delta^+]$, which shows that all the eigenvalues lie within Ω_K ($\alpha_1 = 2$). As highlighted in Subsection III-C, the bound on Δ depends on the choice of network gain parameter β . The change in the upper-bound on Δ with respect to the variation in β is depicted in Figure 10-(b). In addition, for different values of $\Delta \in [-0.45, 0.45]$, the states of leader and follower agents are depicted in Figure 11, which shows that the states of follower agents converge to the states of the leader agent, satisfying a decay rate 2.

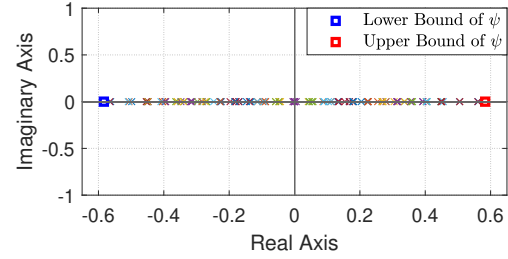


Fig. 9. Eigenvalues of the matrix $\mathcal{L}_{h\delta}$ for $|\Delta| < 0.46$, $\beta = 0.002$. The range of Δ is divided into 20 equal intervals.

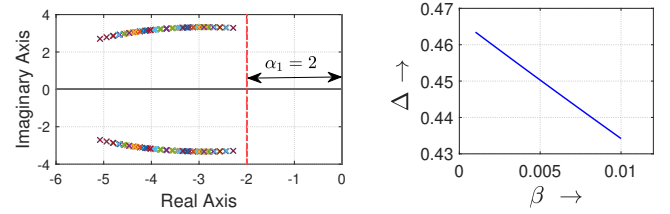


Fig. 10. Left: $\text{fnp}((I_r \otimes E), (I_r \otimes A_y) + \mathcal{L}_h \otimes BK) \in \Omega_K$ for $\alpha_1 = 2$. The range of Δ is divided into 20 equal intervals. Right: β and Δ relation.

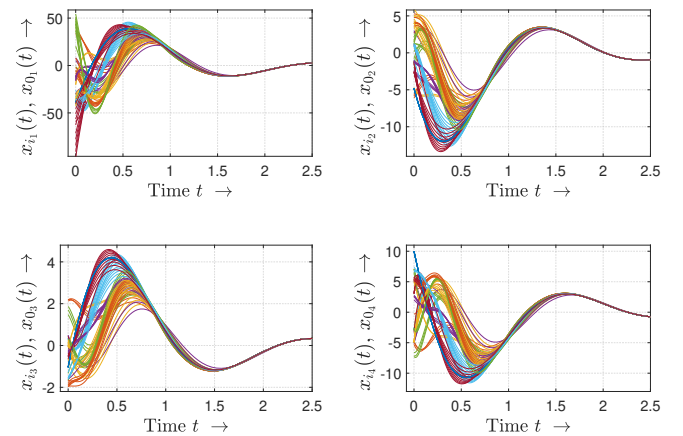


Fig. 11. Synchronization of follower states x_{ij} (solid lines) with leader state x_{0j} (dotted lines) for $\alpha_1 = 2$ for Example 3.

V. CONCLUDING REMARKS

In this article, we have developed a reduced-order observer-based distributed feedback control architecture for an SMAS

under the assumption that both cooperative and antagonistic interactions exist among the agents. The proposed control strategy has ensured that the closed-loop system is impulse-free and that the follower states synchronize with the leader state at a prescribed decay rate. Additionally, the proposed control approach has been extended to the case of an uncertain network graph. The effectiveness of the proposed algorithm has been demonstrated through numerical examples, yielding satisfactory results. The advantages of the proposed algorithm can be summarized as follows. The design of the reduced-order observer is numerically efficient, since we have used only orthogonal matrices for the construction of the system matrices. Furthermore, the algorithm offered flexibility in the independent design of the network gain parameters, the controller gain matrix, and the observer gain matrix, which were computed efficiently by solving LMI feasibility problems with existing solvers.

The limitations and further research directions of the proposed work are as follows. We have addressed the synchronization problem for a class of uncertain networks. It would be interesting to extend this work to a broader class of uncertain networks. Moreover, the order of the observer has been reduced to $\text{rank}(E)$. Although it could further be reduced to an order lower than $\text{rank}(E)$, the separate design of observer and controller gains, as established in the proposed observer framework, may not hold. These ideas are currently under investigation. Furthermore, in a networked microgrid system, it is not always easy to exactly identify the microgrids that are behaving antagonistically by observing their real-time power sharing data. Hence, a possible direction for further research is to identify an antagonistically behaving microgrid in the network by observing its real-time power sharing data. In addition, it would be interesting to extend the considered synchronization problem to more general adversarial scenarios, such as involving external disturbances, measurement noise, and cyber-attacks.

APPENDIX A

Proof of Theorem 1: Corresponding to the full order state observer (10), define the following projectors: $P_d := E_z^+ E_z$, $P'_d := I - E_z^+ E_z$, $P_a := E_z E_z^+$ and $P'_a := I - E_z E_z^+$. Then, by defining the following vectors: $\hat{x}_{d_i} := P_d \hat{x}_i$ and $\hat{x}_{a_i} := P'_d \hat{x}_i$, we have: $\hat{x}_i(t) = \hat{x}_{d_i} + \hat{x}_{a_i}$. It then directly follows from [35] and [40] that the solution of (10) is: $\hat{x}_i(t) = \hat{x}_{d_i}(t) + \hat{x}_{a_i}(t)$, where $\hat{x}_{d_i}$ and $\hat{x}_{a_i}$ are the solutions of following set of differential and algebraic equations:

$$\dot{\hat{x}}_{d_i} = Q \begin{bmatrix} F_d & 0 \\ 0 & 0 \end{bmatrix} Q^T \hat{x}_{d_i} + Q \begin{bmatrix} G_d & L_d \\ 0 & 0 \end{bmatrix} u_o, \quad (45a)$$

$$\hat{x}_{a_i} = Q \begin{bmatrix} 0 & 0 \\ F_a & 0 \end{bmatrix} Q^T \hat{x}_{d_i} + Q \begin{bmatrix} 0 & 0 \\ G_a & L_a \end{bmatrix} u_o. \quad (45b)$$

The system matrices of (45) are as in (13). Then, by defining new state variables: $w_{d_i}(t) := Q^T \hat{x}_{d_i}(t)$ and $w_{a_i}(t) := Q^T \hat{x}_{a_i}(t)$, where Q is the orthogonal matrix used in (12), the observer dynamics in (45) can be represented as follows:

$$\begin{aligned} \dot{w}_{d_i} &= \begin{bmatrix} F_d & 0 \\ 0 & 0 \end{bmatrix} w_{d_i} + \begin{bmatrix} G_d \\ 0 \end{bmatrix} u_{s_i} + \begin{bmatrix} L_d \\ 0 \end{bmatrix} y_i, \\ w_{a_i} &= \begin{bmatrix} 0 & 0 \\ F_a & 0 \end{bmatrix} w_{d_i} + \begin{bmatrix} 0 \\ G_a \end{bmatrix} u_{s_i} + \begin{bmatrix} 0 \\ L_a \end{bmatrix} y_i. \end{aligned} \quad (46)$$

Then, partitioning the vectors w_{d_i} and w_{a_i} conformally with the matrices in (46), and representing them as: $w_{d_i} =$

$[\hat{w}_{d_i}^T \quad \bar{w}_{d_i}^T]^T$, $w_{a_i} = [\bar{w}_{a_i}^T \quad \hat{w}_{a_i}^T]^T$, following differential and algebraic equations are obtained from (46):

$$\begin{aligned} \dot{\hat{w}}_{d_i} &= F_d \hat{w}_{d_i} + G_d u_{s_i} + L_d y_i, \quad \dot{\bar{w}}_{d_i} = 0, \\ \bar{w}_{a_i} &= 0, \quad \hat{w}_{a_i} = F_a \hat{w}_{d_i} + G_a u_{s_i} + L_a y_i. \end{aligned} \quad (47)$$

By setting $\bar{w}_{d_i} = 0$ and $\bar{w}_{a_i} = 0$, the relations $\dot{\bar{w}}_{d_i} = 0$ and $\bar{w}_{a_i} = 0$ are satisfied. With this set-up, we have: $w_{d_i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \hat{w}_{d_i}$ and $w_{a_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{w}_{a_i}$. Hence, by denoting $\hat{w}_i = w_{d_i} + w_{a_i}$, we have the relation as in (15). Moreover, by using the definitions of w_{d_i} and w_{a_i} , we can write:

$$\hat{w}_i = Q^T \hat{x}_{d_i} + Q^T \hat{x}_{a_i} = Q^T (P_d + P'_d) \hat{x}_i = Q^T \hat{x}_i.$$

This directly implies that the estimated state can be obtained as follows: $\hat{x}_i = Q \hat{w}_i$.

We now show that $\hat{x}_i(t)$ asymptotically converges to the actual state $x_i(t)$ for $i \in \mathbb{N}_0$. For this, the agent dynamics in (6) are rewritten as follows:

$$\begin{aligned} E \dot{x}_i &= A_y x_i + B u_{s_i} + L y_i - L y_i \\ &= A_l x_i + B u_{s_i} + L y_i. \end{aligned} \quad (48)$$

Pre-multiply the orthogonal matrix Z , used in (10), on both sides of (48). Then, similar to (45), we have the following set of differential and algebraic equations as a solution of (48) (also solution for (6)):

$$\dot{x}_{d_i} = Q \begin{bmatrix} F_d & 0 \\ 0 & 0 \end{bmatrix} Q^T x_{d_i} + Q \begin{bmatrix} G_d & L_d \\ 0 & 0 \end{bmatrix} u_o, \quad (49a)$$

$$x_{a_i} = Q \begin{bmatrix} 0 & 0 \\ F_a & 0 \end{bmatrix} Q^T x_{d_i} + Q \begin{bmatrix} 0 & 0 \\ G_a & L_a \end{bmatrix} u_o, \quad (49b)$$

where $x_i(t) = x_{d_i} + x_{a_i}$ with $x_{d_i} := P_d x_i$ and $x_{a_i} := P'_d x_i$. Define the following new variables: $z_{d_i} := Q^T x_{d_i}$, $z_{a_i} := Q^T x_{a_i}$, and represent them as follows: $z_{d_i} = [\bar{z}_{d_i}^T \quad \hat{z}_{d_i}^T]^T$, $z_{a_i} = [\bar{z}_{a_i}^T \quad \hat{z}_{a_i}^T]^T$ (partitioned conformally with the matrices in (46)). This leads to the following set of differential and algebraic equations from (49):

$$\begin{aligned} \dot{\hat{z}}_{d_i} &= F_d \hat{z}_{d_i} + G_d u_{s_i} + L_d y_i, \quad \dot{\bar{z}}_{d_i} = 0, \\ \bar{z}_{a_i} &= 0, \quad \hat{z}_{a_i} = F_a \hat{z}_{d_i} + G_a u_{s_i} + L_a y_i. \end{aligned} \quad (50)$$

Then, the following set of differential and algebraic equations is obtained from (47) and (50):

$$\dot{\hat{z}}_{d_i} - \dot{\hat{w}}_{d_i} = F_d (\hat{z}_{d_i} - \hat{w}_{d_i}), \quad (51a)$$

$$\dot{\bar{z}}_{d_i} - \dot{\bar{w}}_{d_i} = 0, \quad (51b)$$

$$\bar{z}_{a_i} - \bar{w}_{a_i} = 0, \quad (51c)$$

$$\hat{z}_{a_i} - \hat{w}_{a_i} = F_a (\hat{z}_{d_i} - \hat{w}_{d_i}). \quad (51d)$$

Since we have set $\bar{w}_{d_i} = 0$ and $\bar{w}_{a_i} = 0$, from (51b) and (51c), following set of relations are obtained: $\dot{\bar{z}}_{d_i} = 0$ and $\bar{z}_{a_i} = \bar{w}_{a_i} = 0$, respectively. The vector $\bar{z}_{d_i} = 0$, which can be seen as follows. Using the definition of matrices P_d and P'_d , and the fact that $E_z^+ E_z E_z^+ = E_z^+$, it is easy to see that $x_{d_i}^T x_{a_i} = 0$. Further, since Q is orthogonal:

$$\begin{aligned} x_{d_i}^T x_{a_i} &= z_{d_i}^T Q^T Q z_{a_i} = [\bar{z}_{d_i}^T \quad \hat{z}_{d_i}^T] \begin{bmatrix} \bar{z}_{a_i} \\ \hat{z}_{a_i} \end{bmatrix} \\ &= \bar{z}_{d_i}^T \bar{z}_{a_i} + \hat{z}_{d_i}^T \hat{z}_{a_i}. \end{aligned} \quad (52)$$

According to (51c) and (51d), we have: $\bar{z}_{a_i} = 0$ and $\bar{z}_{a_i} \neq 0$. Hence, to satisfy $x_{d_i}^T x_{a_i} = 0$, it follows from (52) that $\bar{z}_{d_i} = 0$, which implies $\bar{z}_{d_i} = 0$. It can be shown (similar to [35, Theorem 3]) that $\text{fnp}(E, A_l) = \text{eig}(F_d)$. Since L is designed such that $\text{fnp}(E, A_l) \in \Omega_L$, this implies $\text{eig}(F_d) \in \Omega_L$. Hence, $\lim_{t \rightarrow \infty} \|\bar{z}_{d_i} - \hat{w}_{d_i}\|_2 = 0$. This implies, $\lim_{t \rightarrow \infty} \|\bar{z}_{a_i} - \hat{w}_{a_i}\|_2 = 0$. Further, since $w_{d_i} = [\hat{w}_{d_i}^T \ 0^T]^T$, $z_{d_i} = [\bar{z}_{d_i}^T \ 0^T]^T$ and Q is orthogonal, we have the following relations: $\lim_{t \rightarrow \infty} \|z_{d_i} - w_{d_i}\|_2 = \lim_{t \rightarrow \infty} \|Qz_{d_i} - Qw_{d_i}\|_2 = \lim_{t \rightarrow \infty} \|x_{d_i} - \hat{x}_{d_i}\|_2 = 0$. Similarly, it is easy to see that $\lim_{t \rightarrow \infty} \|x_{a_i} - \hat{x}_{a_i}\|_2 = 0$. Hence, $\hat{x}_i = \hat{x}_{d_i} + \hat{x}_{a_i}$ will converge to the state $x_i = x_{d_i} + x_{a_i}$ asymptotically, for all $i \in \mathbb{N}_0$. This completes the proof.

APPENDIX B

Proof of Theorem 2: Before starting the proof, we introduce the following vectors, which are used in the proof:

$$\begin{aligned} \varepsilon &= [\varepsilon_1^T \ \varepsilon_2^T \ \cdots \ \varepsilon_r^T]^T, \quad \tilde{s}_0 = [\tilde{s}_0^T \ \tilde{s}_0^T \ \cdots \ \tilde{s}_0^T]^T, \\ \tilde{s} &= [\tilde{s}_1^T \ \tilde{s}_2^T \ \cdots \ \tilde{s}_r^T]^T, \quad \tilde{x}_d := [\tilde{x}_{d_1}^T \ \tilde{x}_{d_2}^T \ \cdots \ \tilde{x}_{d_r}^T]^T, \\ \tilde{x}_a &= [\tilde{x}_{a_1}^T \ \tilde{x}_{a_2}^T \ \cdots \ \tilde{x}_{a_r}^T]^T, \quad \tilde{x} := [\tilde{x}_1^T \ \tilde{x}_2^T \ \cdots \ \tilde{x}_r^T]^T, \quad (53) \\ \mathbf{z}_d &= [z_{d_1}^T \ z_{d_2}^T \ \cdots \ z_{d_r}^T]^T \quad \mathbf{w}_d = [w_{d_1}^T \ w_{d_2}^T \ \cdots \ w_{d_r}^T]^T, \\ \tau &= [\tau_1^T \ \tau_2^T \ \cdots \ \tau_r^T]^T, \quad \bar{\tau}_0 = [\tau_0^T \ \tau_0^T \ \cdots \ \tau_0^T]^T. \end{aligned}$$

Since $\mathcal{L}_g$ is a symmetric matrix, there exists an orthogonal matrix $W \in \mathbb{R}^{r \times r}$ such that:

$$W^T \mathcal{L}_g W = \Lambda_g = \text{diag}\{\lambda_1(\mathcal{L}_g), \lambda_2(\mathcal{L}_g), \dots, \lambda_r(\mathcal{L}_g)\}. \quad (54)$$

Using W we define the following new state variables: $\varepsilon := (W^T \otimes I_n)\delta$, $\tilde{s}_0 := (W^T \otimes I_n)\tilde{x}_0$ and $\tilde{s} := (W^T \otimes I_n)\tilde{x}$. Then, the synchronization error dynamics (23) is transformed into the following error dynamics:

$$\begin{aligned} (I_r \otimes E)\dot{\varepsilon} &= [(I_r \otimes A_y) + (W^T \mathcal{L}_g W \otimes BK)]\varepsilon - \\ &\quad (W^T \mathcal{L}_g W \otimes BK)\tilde{s} + (W^T \mathcal{L}_g W \otimes BK)\tilde{s}_0 \\ &= [(I_r \otimes A_y) + (\Lambda_g \otimes BK)]\varepsilon - (\Lambda_g \otimes BK)\tilde{s} \\ &\quad + (\Lambda_g \otimes BK)\tilde{s}_0. \end{aligned} \quad (55)$$

Now consider the relations in (45) and (49). By subtracting (45a) from (49a) (both sides) and (45b) from (49b), we have:

$$\dot{\tilde{x}}_{d_i} = Q\bar{F}_d Q^T \tilde{x}_{d_i}, \quad \tilde{x}_{a_i} = Q\bar{F}_a Q^T \tilde{x}_{a_i}, \quad (56)$$

where $\bar{F}_d = \begin{bmatrix} F_d & 0 \\ 0 & 0 \end{bmatrix}$ and $\bar{F}_a = \begin{bmatrix} 0 & 0 \\ F_a & 0 \end{bmatrix}$. Since $\tilde{x}_i = \tilde{x}_{d_i} + \tilde{x}_{a_i}$, and hence, $\tilde{x} = \tilde{x}_d + \tilde{x}_a$, the relations in (56) can compactly be written as:

$$\dot{\tilde{x}}_d = (I_r \otimes Q\bar{F}_d Q^T)\tilde{x}_d, \quad \dot{\tilde{x}}_a = (I_r \otimes Q\bar{F}_a Q^T)\tilde{x}_a. \quad (57)$$

Then, the term $(\Lambda_g \otimes BK)\tilde{s}$ that appears in (55), can be written as follows, using the definition of $\tilde{s}$:

$$\begin{aligned} (\Lambda_g \otimes BK)\tilde{s} &= (\Lambda_g \otimes BK)(W^T \otimes I_n)\tilde{x} \\ &= (\Lambda_g W^T \otimes BK)(\tilde{x}_d + \tilde{x}_a) \\ &= (\Lambda_g W^T \otimes BKQ)(I_r \otimes Q^T)(\tilde{x}_d + \tilde{x}_a) \\ &= (\Lambda_g W^T \otimes BKQ)(I_{nr} + (I_r \otimes \bar{F}_a)) \\ &\quad (I_r \otimes Q^T)\tilde{x}_d. \end{aligned} \quad (58)$$

Recall from the proof of Theorem 1 that $w_{d_i} = Q^T \hat{x}_{d_i}$ and $z_{d_i} = Q^T x_{d_i}$. Hence, $(I_r \otimes Q^T)\tilde{x}_d = (I_r \otimes Q^T)(\mathbf{x}_d - \hat{\mathbf{x}}_d) = \mathbf{z}_d - \mathbf{w}_d$. Using this relation, (58) becomes:

$$(\Lambda_g \otimes BK)\tilde{s} = (\Lambda_g W^T \otimes BKQ)(I_r \otimes J)(\mathbf{z}_d - \mathbf{w}_d), \quad (59)$$

where $J = I_n + \bar{F}_a$. It follows from the proof of Theorem 1 that $z_{d_i} = [\bar{z}_{d_i}^T \ 0^T]^T$ and $w_{d_i} = [\hat{w}_{d_i}^T \ 0^T]^T$. Now, define an error vector $\tau_i := \bar{z}_{d_i} - \hat{w}_{d_i}$. Then, it can be noticed that: $\mathbf{z}_d - \mathbf{w}_d = (I_r \otimes \begin{bmatrix} I_q \\ 0 \end{bmatrix})\tau$. Write the matrix $BKQJ = [B_{q_1} \ B_{q_2}]$, where the partition is conformal with the partition in vector z_{d_i} . Then, (59) becomes:

$$\begin{aligned} (\Lambda_g \otimes BK)\tilde{s} &= (\Lambda_g W^T \otimes BKQJ)(I_r \otimes \begin{bmatrix} I_q \\ 0 \end{bmatrix})\tau \\ &= (\Lambda_g W^T \otimes B_{q_1})\tau. \end{aligned} \quad (60)$$

Similarly, the term $(\Lambda_g \otimes BK)\tilde{s}_0$ can be represented as:

$$(\Lambda_g \otimes BK)\tilde{s}_0 = (\Lambda_g W^T \otimes B_{q_1})\bar{\tau}_0. \quad (61)$$

Then, using (60) and (61) in (55), the transformed synchronization error dynamics becomes:

$$\begin{aligned} (I_r \otimes E)\dot{\varepsilon} &= [(I_r \otimes A_y) + (\Lambda_g \otimes BK)]\varepsilon \\ &\quad - (\Lambda_g W^T \otimes B_{q_1})\tau + (\Lambda_g W^T \otimes B_{q_1})\bar{\tau}_0. \end{aligned} \quad (62)$$

Further, it follows from (51a) that $\dot{\tau}_i = F_d \tau_i$, for all $i \in \mathbb{N}_0$. Hence, we have the following relations: $\dot{\tau} = (I_r \otimes F_d)\tau$ and $\dot{\bar{\tau}}_0 = (I_r \otimes F_d)\bar{\tau}_0$. By defining a global error vector $\xi := [\bar{\tau}_0^T \ \tau^T \ \varepsilon^T]^T \in \mathbb{R}^{2rq+rn}$, we have the following global error dynamics:

$$\mathbf{E}_\xi \dot{\xi}(t) = \mathbf{A}_\xi \xi(t), \quad \forall i \in \mathbb{N}, \quad (63)$$

where the matrices $\mathbf{E}_\xi$ and $\mathbf{A}_\xi$ are:

$$\begin{aligned} \mathbf{E}_\xi &= \begin{bmatrix} I_{rq} & 0 & 0 \\ 0 & I_{rq} & 0 \\ 0 & 0 & I_r \otimes E \end{bmatrix}, \\ \mathbf{A}_\xi &= \begin{bmatrix} I_r \otimes F_d & 0 & 0 \\ 0 & I_r \otimes F_d & 0 \\ (\Lambda_g W^T \otimes B_{q_1}) & (-\Lambda_g W^T \otimes B_{q_1}) & (I_r \otimes A_y) + (\Lambda_g \otimes BK) \end{bmatrix}. \end{aligned}$$

The finite poles of (63) are equal to the roots of polynomial $\det(s\mathbf{E}_\xi - \mathbf{A}_\xi)$. Using the definition of $\mathbf{E}_\xi$ and $\mathbf{A}_\xi$, and determinant properties, we have:

$$\begin{aligned} \det(s\mathbf{E}_\xi - \mathbf{A}_\xi) &= (\det(sI_q - F_d))^{2r} \\ &\quad \prod_{i=1}^r (\det(sE - A_y - \lambda_i(\mathcal{L}_g)BK)). \end{aligned}$$

Then, by denoting a set Λ_{F_d} as the set of eigenvalues of matrix F_d with their algebraic multiplicity $2r$, we have the following set theoretic relation:

$$\text{fnp}(\mathbf{E}_\xi, \mathbf{A}_\xi) = \Lambda_{F_d} \cup (\cup_{i=1}^r \text{fnp}(E, A_y + \lambda_i(\mathcal{L}_g)BK)).$$

By considering the assumptions in (24), the solution $\xi(t)$ of (63) satisfies: $\|\xi(t)\|_2 \leq \kappa e^{-\alpha_1 t} \|\xi(0)\|_2$, where α_1 is the decay rate and κ is a non-zero constant [11, Chapter 3]. Moreover, since $\alpha_2 > \alpha_1 > 0$ ($\Omega_L \subset \Omega_K$), the vector $\lim_{t \rightarrow \infty} \|\xi\|_2 = 0$, with a minimum decay rate α_1 . This implies: $\lim_{t \rightarrow \infty} \|\tau\|_2 = \lim_{t \rightarrow \infty} \|\bar{\tau}_0\|_2 = \lim_{t \rightarrow \infty} \|\varepsilon\|_2 = 0$, with a minimum decay rate α_1 . However, since $\dot{\tau} = (I_r \otimes F_d)\tau$, $\dot{\bar{\tau}}_0 = (I_r \otimes F_d)\bar{\tau}_0$, and $\text{eig}(F_d) \in \Omega_L$, the vectors τ , $\bar{\tau}_0$ converge to zero satisfying a minimum decay rate $\alpha_2 > \alpha_1$. Then, it directly

follows from the proof of Theorem 1 (last paragraph) that the first statement holds. Since $\lim_{t \rightarrow \infty} \|\varepsilon\|_2 = 0$, with a minimum decay rate α_1 , and $\varepsilon = (W^T \otimes I_n)\delta$, we have: $\lim_{t \rightarrow \infty} \|\varepsilon\|_2 = \lim_{t \rightarrow \infty} \|(W^T \otimes I_n)\delta\|_2 = \lim_{t \rightarrow \infty} \|\delta\|_2 = 0$. Hence, δ_i for all $i \in \mathbb{N}$ converge to zero with a minimum decay rate α_1 . This implies the second statement also holds. This completes the proof.

APPENDIX C

To prove Theorem 4, we need some preliminaries on polynomial matrices, which are taken from [41, Chapter 7]. Let $G(s)$ be a polynomial matrix of size $p \times m$ with entries that are polynomials in s with real coefficients. The degree of i^{th} row of $G(s)$, denoted as d_{r_i} , is the degree of *highest* degree polynomial in the i^{th} row. By defining a diagonal polynomial matrix as follows: $S_d(s) := \text{diag}(s^{d_{r_1}}, s^{d_{r_2}}, \dots, s^{d_{r_p}})$, every polynomial matrix can be represented as follows

$$G(s) = S_d(s)G_r + \hat{G}(s), \quad (64)$$

where $G_r \in \mathbb{R}^{p \times m}$ is a real matrix, referred to as *highest row degree coefficient matrix* of $G(s)$. Then, we have the following proposition, which is directly taken from [41, Section 7.2].

Proposition 1: Assume that a $p \times p$ size polynomial matrix $G(s)$ is *row-reduced*, that is, the rank of the highest row degree coefficient matrix G_r is p . Then, the following relation holds: $\det(G(s)) = \det(G_r)s^\nu + g(s)$, where $\nu = \sum_{i=1}^p d_{r_i}$ and $g(s)$ is a polynomial with degree strictly less than ν . Using this result, we prove the following result.

Lemma 1: Consider the system matrices that appear in the closed-loop system dynamics (34). Then, we have:

$$\deg \left\{ \det \begin{bmatrix} sE - A_y & -\lambda_i(\mathcal{L}_g)BK_1 \\ -L_dC & sI_q - F_d - \lambda_i(\mathcal{L}_g)G_dK_1 \end{bmatrix} \right\} = 2q.$$

Proof: For a matrix E , there exist two orthogonal matrices (P', Q') , such that $Q'EP' = \begin{bmatrix} \Sigma'_q & 0 \\ 0 & 0 \end{bmatrix}$ where Σ'_q is a diagonal matrix with diagonal entries that are the non-zero singular values of E . Using P' and Q' , we have the following partitioning of the matrices, conformal with $\begin{bmatrix} \Sigma'_q & 0 \\ 0 & 0 \end{bmatrix}$: $Q'A_yP' = \begin{bmatrix} A'_{y11} & A'_{y12} \\ A'_{y21} & A'_{y22} \end{bmatrix}$, $QB' = \begin{bmatrix} B'_1 \\ B'_2 \end{bmatrix}$, $CP' = [C'_1 \ C'_2]$. Then, we obtain the relation given in (65):

$$\begin{aligned} & \deg \left\{ \det \begin{bmatrix} sE - A_y & -\lambda_i(\mathcal{L}_g)BK_1 \\ -L_dC & sI_q - F_d - \lambda_i(\mathcal{L}_g)G_dK_1 \end{bmatrix} \right\} \\ &= \deg \left\{ \det \begin{bmatrix} Q' & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sE - A_y & -\lambda_i(\mathcal{L}_g)BK_1 \\ -L_dC & sI_q - F_d - \lambda_i(\mathcal{L}_g)G_dK_1 \end{bmatrix} \begin{bmatrix} P' & 0 \\ 0 & I \end{bmatrix} \right\} \\ &= \deg \{ \det(G(s)) \}, \end{aligned} \quad (65)$$

where: $G(s) := \begin{bmatrix} s\Sigma'_q - A'_{y11} & -A'_{y12} & \lambda_i(\mathcal{L}_g)B'_1K_1 \\ -A'_{y21} & -A'_{y22} & \lambda_i(\mathcal{L}_g)B'_2K_1 \\ -L_dC'_1 & -L_dC'_2 & sI_q - F_d + \lambda_i(\mathcal{L}_g)G_dK_1 \end{bmatrix}$. Note that the polynomial matrix $G(s)$ can be represented as (64), where:

$$\begin{aligned} S_d(s) &:= \begin{bmatrix} sI_q & 0 & 0 \\ 0 & I_{n-q} & 0 \\ 0 & 0 & sI_q \end{bmatrix}, \quad G_r = \begin{bmatrix} \Sigma'_q & 0 & 0 \\ -A'_{y21} & -A'_{y22} & \lambda_i B'_2 K_1 \\ 0 & 0 & I_q \end{bmatrix} \\ d_{r_i} &= 1, \quad \text{for } i = 1, 2, \dots, q \\ d_{r_i} &= 0, \quad \text{for } i = q+1, q+2, \dots, n \\ d_{r_i} &= 1, \quad \text{for } i = n+1, n+2, \dots, n+q. \end{aligned} \quad (66)$$

Recall that the matrix pair (E, A_y) is impulse-free, and hence, it follows from [11, Theorem 7.3] that A'_{y22} is nonsingular. This implies the matrix G_r is full-rank, and hence, the polynomial matrix $G(s)$ is row-reduced. Further, according to (66), the quantity: $\nu = \sum_{i=1}^{n+q} d_{r_i} = 2q$. Then, the result directly follows from Proposition 1. ■

Proof of Theorem 4: The closed-loop state trajectories are free of impulses if and only if the following condition holds [11, Theorem 7.1]: $\deg\{\det(sE_{cl} - A_{cl})\} = \text{rank}(E_{cl})$, where E_{cl} and A_{cl} are as in †. Applying the property of determinant of a block triangular matrix [42, Section 0.9.4], we obtain:

$$\deg\{\det(sE_{cl} - A_{cl})\} = \deg\{\det(sE_{cl_{11}} - A_{cl_{11}})\} + \deg\{\det(sE_{cl_{22}} - A_{cl_{22}})\}. \quad (67)$$

We first evaluate the first term in the right-hand side of (67):

$$\begin{aligned} & \deg\{\det(sE_{cl_{11}} - A_{cl_{11}})\} \\ &= \deg \left\{ \det \begin{bmatrix} I_r \otimes (sE - A_y) & 0 & 0 & 0 \\ -I_r \otimes L_dC & I_r \otimes (sI_q - F_d) & 0 & 0 \\ -I_r \otimes L_aC & -I_r \otimes F_a & I_{r_q} & 0 \\ 0 & -I_r \otimes Q_1 & -I_r \otimes Q_2 & I_{r_n} \end{bmatrix} \right\} \\ &= \deg\{\det(I_r \otimes (sE - A_y))(I_r \otimes (sI_q - F_d))I_{r_q}I_{r_n}\} \\ &= \deg\{\det(I_r \otimes (sE - A_y))\} + \deg\{\det(I_r \otimes (sI_q - F_d))\}. \end{aligned} \quad (68)$$

Recall from (6) that the pair (E, A_y) is impulse-free, and hence, $\deg\{\det(sE - A_y)\} = q$, where $q = \text{rank}(E)$ [11, Theorem 7.1]. Then, using the Kronecker product property: $\deg\{\det(I_r \otimes (sE - A_y))\} = rq$. Furthermore, it follows from the proof of Theorem 1 that F_d has q number of eigenvalues, and hence, we have: $\deg\{\det(sI_r \otimes (I_q - F_d))\} = rq$. Thus, $\deg\{\det(sE_{cl_{11}} - A_{cl_{11}})\} = 2rq$, according to (68).

We now evaluate the second term of the right-hand side of (67):

$$\begin{aligned} & \deg\{\det(sE_{cl_{22}} - A_{cl_{22}})\} = \\ & \deg \left\{ \det \begin{bmatrix} I_{r_n} & 0 & -I_r \otimes Q_1 & -I_r \otimes Q_2 \\ -\mathcal{L}_g \otimes BK & I_r \otimes (sE - A_y) & 0 & 0 \\ -\mathcal{L}_g \otimes G_dK & -I_r \otimes L_dC & I_r \otimes (sI_q - F_d) & 0 \\ -\mathcal{L}_g \otimes G_aK & -I_r \otimes L_aC & -I_r \otimes F_a & I_{r_q} \end{bmatrix} \right\}. \end{aligned}$$

Using determinant of a block matrix property [42], we have:

$$\begin{aligned} & \deg\{\det(sE_{cl_{22}} - A_{cl_{22}})\} \\ &= \deg \left\{ \det(I_{r_n}) \det \left(\begin{bmatrix} I_r \otimes (sE - A_y) & 0 & 0 \\ -I_r \otimes L_dC & I_r \otimes (sI_q - F_d) & 0 \\ -I_r \otimes L_aC & -I_r \otimes F_a & I_{r_q} \end{bmatrix} - \begin{bmatrix} -\mathcal{L}_g \otimes BK \\ -\mathcal{L}_g \otimes G_dK \\ -\mathcal{L}_g \otimes G_aK \end{bmatrix} I_{r_n}^{-1} \begin{bmatrix} 0 & -I_r \otimes Q_1 & -I_r \otimes Q_2 \end{bmatrix} \right) \right\}. \end{aligned} \quad (69)$$

Further, using the definition of $K := [K_1 \ 0]Q^T$, as defined in (25), and the property: $Q_i^T Q_i = I$ and $Q_i^T Q_j = 0$ for $i \neq j$, we have:

$$\begin{aligned} & \deg\{\det(sE_{cl_{22}} - A_{cl_{22}})\} = \\ & \deg \left\{ \det \begin{bmatrix} I_r \otimes (sE - A_y) & -\mathcal{L}_g \otimes BK_1 & 0 \\ -I_r \otimes L_dC & I_r \otimes (sI_q - F_d) - \mathcal{L}_g \otimes G_dK_1 & 0 \\ -I_r \otimes L_aC & -I_r \otimes F_a - \mathcal{L}_g \otimes G_aK_1 & I_{r_q} \end{bmatrix} \right\}. \end{aligned} \quad (70)$$

$$E_{cl} = \left[\begin{array}{c|c} E_{cl11} & E_{cl12} \\ \hline E_{cl21} & E_{cl22} \end{array} \right] = \left[\begin{array}{cccc|cccc} I_r \otimes E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_r \otimes I_q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_r \otimes E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_r \otimes I_q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right],$$

$$A_{cl} = \left[\begin{array}{c|c} A_{cl11} & A_{cl12} \\ \hline A_{cl21} & A_{cl22} \end{array} \right] = \left[\begin{array}{cccc|cccc} I_r \otimes A_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_r \otimes L_d C & I_r \otimes F_d & 0 & 0 & 0 & 0 & 0 & 0 \\ I_r \otimes L_a C & I_r \otimes F_a & -I_{rq} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_r \otimes Q_1 & I_r \otimes Q_2 & -I_{rn} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -I_{rn} & 0 & I_r \otimes Q_1 & I_r \otimes Q_2 \\ 0 & 0 & 0 & 0 & \mathcal{L}_g \otimes BK & \mathcal{L}_g \otimes BK & I_r \otimes A_y & 0 \\ 0 & 0 & 0 & 0 & \mathcal{L}_g \otimes G_d K & \mathcal{L}_g \otimes G_d K & I_r \otimes L_d C & I_r \otimes F_d \\ 0 & 0 & 0 & 0 & \mathcal{L}_g \otimes G_a K & \mathcal{L}_g \otimes G_a K & I_r \otimes L_a C & I_r \otimes F_a \end{array} \right]. \quad (\dagger)$$

Using the orthogonal matrix W , defined in (54), define an orthogonal matrix $\Gamma := \text{blkdiag}\{W \otimes I_n, W \otimes I_q, W \otimes I_q\}$. Then, (70) can be written as:

$$\begin{aligned} & \deg\{\det(sE_{cl22} - A_{cl22})\} \\ &= \deg\left\{\det \Gamma^T \begin{bmatrix} I_r \otimes (sE - A_y) & -\mathcal{L}_g \otimes BK_1 & 0 \\ -I_r \otimes L_d C & I_r \otimes (sI_q - F_d) - \mathcal{L}_g \otimes G_d K_1 & 0 \\ -I_r \otimes L_a C & -I_r \otimes F_a - \mathcal{L}_g \otimes G_a K_1 & I_{rq} \end{bmatrix} \Gamma\right\} \\ &= \deg\left\{\prod_{i=1}^r \det \begin{bmatrix} sE - A_y & -\lambda_i(\mathcal{L}_g)BK_1 & 0 \\ -L_d C & sI_q - F_d - \lambda_i(\mathcal{L}_g)G_d K_1 & 0 \\ -L_a C & -F_a - \lambda_i(\mathcal{L}_g)G_a K_1 & I_q \end{bmatrix}\right\}. \quad (71) \end{aligned}$$

Denote $J_i(s) = \det \begin{bmatrix} sE - A_y & -\lambda_i(\mathcal{L}_g)BK_1 & 0 \\ -L_d C & sI_q - F_d - \lambda_i(\mathcal{L}_g)G_d K_1 & 0 \\ -L_a C & -F_a - \lambda_i(\mathcal{L}_g)G_a K_1 & I_q \end{bmatrix}$, for $i = 1, 2, \dots, r$. Then, using the determinant property of a block triangular matrix, we have:

$$\det J_i(s) = \det \begin{bmatrix} sE - A_y & -\lambda_i(\mathcal{L}_g)BK_1 \\ -L_d C & sI_q - F_d - \lambda_i(\mathcal{L}_g)G_d K_1 \end{bmatrix}. \quad (72)$$

Using the relation (72) in (71), following relation is obtained:

$$\begin{aligned} & \deg\{\det(sE_{cl22} - A_{cl22})\} \\ &= \deg\left\{\prod_{i=1}^r \det \begin{bmatrix} sE - A_y & -\lambda_i(\mathcal{L}_g)BK_1 \\ -L_d C & sI_q - F_d - \lambda_i(\mathcal{L}_g)G_d K_1 \end{bmatrix}\right\} \\ &= \sum_{i=1}^r \deg\left\{\det \begin{bmatrix} sE - A_y & -\lambda_i(\mathcal{L}_g)BK_1 \\ -L_d C & sI_q - F_d - \lambda_i(\mathcal{L}_g)G_d K_1 \end{bmatrix}\right\}. \quad (73) \end{aligned}$$

It then directly follows from Lemma 1 and the relation (73) that: $\deg\{\det(sE_{cl22} - A_{cl22})\} = r(2q) = 2rq$. Thus, (68) can be expressed as $\deg\{\det(sE_{cl} - A_{cl})\} = 2rq + 2rq = 4rq$, which is equal to $\text{rank}(E_{cl})$, hence the result follows.

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